Positive Partial Transpose criterion in Symplectic geometry

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Introduction

The positive partial transpose (PPT) criterion, also known as the Peres-Horodecki criterion, is an entanglement criterion originally introduced for discrete variable systems [1]. It states that if ρ_{AB} is a separable state of a bipartite system, then $\rho_{AB}^{T_B} \geq 0$, that is, the partial transpose of ρ_{AB} is a valid quantum state. Although it gives only a necessary condition for separability in general, its failure to be sufficient is related to entanglement distillation: an entangled state which satisfies PPT cannot be distilled [2]. This criterion has been generalized to continuous variable (CV) systems [3]. In terms of Wigner functions, the partial transpose of a CV state corresponds to inverting the momentum coordinates of the second part,

$$W_{\rho^{T_B}}(q_A, p_A, q_B, p_B) = W_\rho(q_A, p_A, q_B, -p_B). \tag{1}$$

It has been applied to Gaussian states to investigate various properties related to entanglement [4].

In addition to Wigner function, *symplectic geometry* has been applied to investigate quantum phase spaces [5]. Its symplectic structure

$$\omega = \sum_i dq^i \wedge dp_i \tag{2}$$

originates from classical mechanics, and is used in both semiclassical quantization and the Wigner-Weyl transform, which the Wigner function is based on. Here, we investigate the PPT criterion by using symplectic geometry, in order to give a geometric picture on the entanglement of bipartite systems.

Main Result

First, we describe partial transposition in a coordinate-free way. Consider a bipartite CV system, and let (E_A, ω_A) and (E_B, ω_B) be the phase spaces, viewed as symplectic vector spaces, of the two subsystems, then partial transposition on $E = E_A \oplus E_B$ corresponds to a linear symplectomorphism

$$T_B : (E, \omega_+) \to (E, \omega_-) \tag{3}$$

up to a local linear symplectomorphism, where $\omega_\pm = \omega_A \pm \omega_B$. Equivalently, it corresponds to changing the symplectic structure of the composite system from ω_+ to ω_-, so that a state satisfies PPT if and only if its Wigner function is a valid state with respect to ω_-. If the state is parametrized by vector subspaces, we can also describe this using symplectic complements.

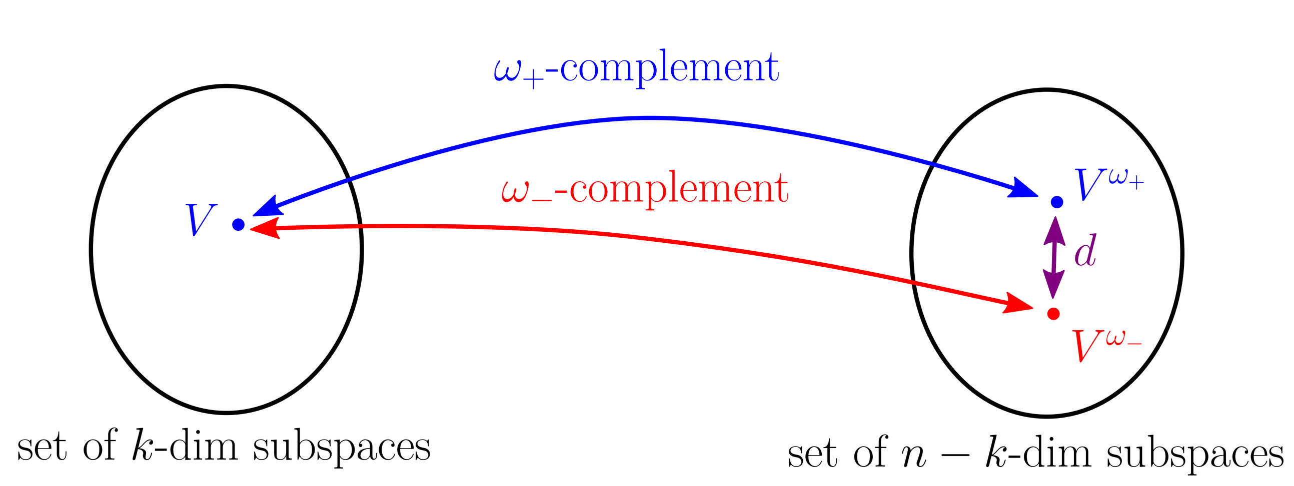

Figure 1: Partial transpose may break the original ω_+-complement relation of the parameters of the state, and d is a candidate for entanglement meaure.

Generalization to Spin phase spaces

In addition to the usual phase space $\mathbb{R}^{2N}$ for CV systems, states of a spin-s system can also be described by Wigner functions on a phase space S^2 [6] with symplectic structure

$$\omega = s \sin\theta \, d\theta \wedge d\phi \, . \tag{4}$$

When s is large, the symplectic structure locally look like that of a single-mode CV system, while when $s = \frac{1}{2}$, it coincides with that of the projective Hilbert space.

For bipartite system involving the S^2 phase space, composition of subsystems can be described by product of symplectic manifolds, and partial transposition by changing the symplectic structure of the composite system $M_A \times M_B$ from ω_+ to ω_-.

PPT of Lagrangian subspaces

Consider the case of n-dimensional subspaces of the phase space $E \simeq \mathbb{R}^{2n}$. Let $L \in \mathrm{Lag}(E, \omega_+)$ be a Lagrangian subspace. It can be shown that L can be written as $L = L_A \oplus L_B$, where $L_i \in \mathrm{Lag}(E_i)$, if and only if $L \in \mathrm{Lag}(E, \omega_-)$. Intuitively, this means that a Lagrangian subspace is "separable" if and only if its partial transpose is Lagrangian. This can be generalized to Lagrangian submanifolds of E, which links the PPT criterion to classical integrable systems, and also to the entanglement of semiclassical quantum states [7].

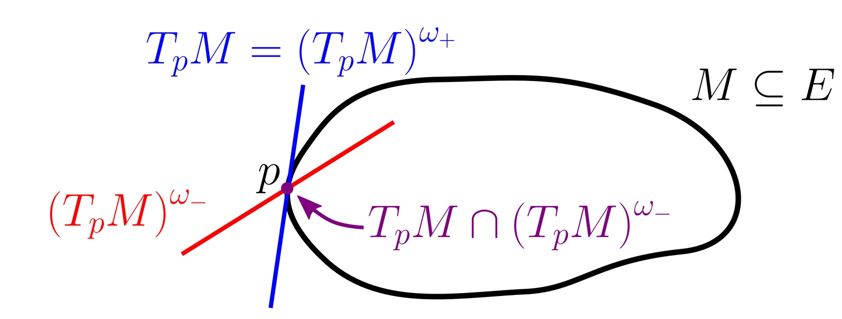

Figure 2: The tangent space of a Lagrangian submanifold may fail to be Lagrangian after partial transpose. The degeneracy $d = \dim T_pM \cap (T_pM)^{\omega_-}$ can be used to measure entanglement.

PPT of Bipartite cat states

The general Wigner function of a bipartite cat state consists of two Gaussians with a interference term at the midpoint proportional to

$$e^{-\|\xi\|^2} \cos(\omega_+(\xi, v)), \tag{5}$$

where ξ is the phase space coordinate vector and v is the vector connecting the centers of the two Gaussians. Note that the hyperplane of equal phase is the symplectic complement of v. The positivity of the partial transpose has a geometrical interpretation in this case.

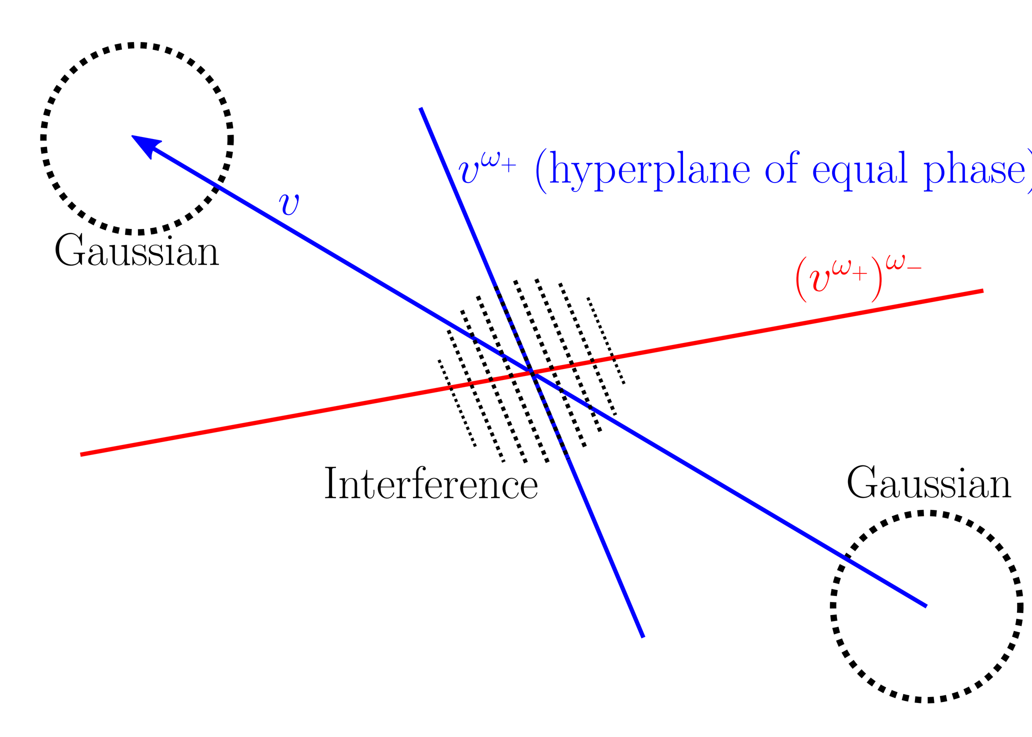

Figure 3: Schmetic diagram of the Wigner function of a bipartite cat state. If there are no Gaussians on the line $(v^{\omega_+})^{\omega_-}$, then the Wigner function after partial transpose contains an isolated interference term, so the state violates PPT.

This can be generalized to any mixture of superpositions of coherent states with all Gaussians and interferences far apart. Moreover, for a non-PPT state of this type, we can contruct a corresponding PPT state by mixing it with some coherent states so that on each $(v^{\omega_+})^{\omega_-}$, there are two Gaussians at the appropriate positions.

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