Non-Abelian fracton order from gauging a mixture of subsystem and global symmetries

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Motivation

2D non-Abelian anyons

- Nonlocal degrees of freedom
- Fault-tolerant quantum computation
- Modular tensor category

3D fracton order

- Immobile excitations
- Suitable for quantum memory
- Related to tensor gauge theory

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Previous constructions

- Cage-net models (Prem, Huang, Song, Hermele)
- Layer constructions (Vijay, Fu; Williamson, Cheng; Song, Prem, Huang, Martin-Delgado)
- Gauging bilayer/permutation symmetries (Prem, Williamson; Bulmash, Barkeshli)
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Gauging: promoting a nonlocal symmetry to a local one by introducing gauge DOF.

Outline

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Part I: Gauging

Gauging a global symmetry

Gauging a subsystem symmetry

Our construction: gauging a mixture of subsystem and global symmetries

Part II: Algebra of operators

The algebra of Kitaev's quantum double model

The algebra of our fracton model

Gauging global $\mathbb{Z}_2$ spin-flip symmetry

Starting point: Transverse-field Ising model on a 3D cubic lattice

$$H = -J \sum_{\text{links}} \begin{array}{c} Z \\ | \\ Z \end{array} - h \sum_{\text{sites}} X \tag{1}$$

Gauging global $\mathbb{Z}_2$ spin-flip symmetry

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$$H = -J \sum_{\text{links}} \begin{matrix} Z \\ | \\ Z \end{matrix} - h \sum_{\text{sites}} X \tag{1}$$

1. Introduce a gauge spin on each link, and modify:

$$\begin{matrix} Z \\ | \\ Z \end{matrix} \rightarrow \begin{matrix} Z \\ | \\ Z \\ | \\ Z \end{matrix} \tag{2}$$

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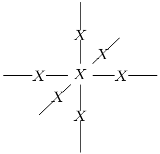
2. Enforce gauge symmetry on each vertex

Constraint:  $|\psi\rangle = |\psi\rangle$ (3)

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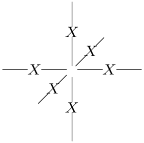
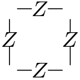
3. Add the gauge flux terms

$$H_{\text{flux}} = -\sum_{\text{plaquettes}} \begin{array}{ccc} & -Z- & \\ | & & | \\ Z & & Z \\ | & & | \\ & -Z- & \end{array} \tag{4}$$

Excitations of the 3D toric code

$H = -\sum_{\text{stars}}$ 

$-\sum_{\text{plaquettes}}$ 

(5)

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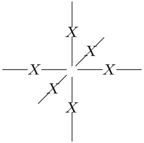
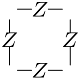
$$H = -\sum_{\text{stars}}$$ 

$$- \sum_{\text{plaquettes}}$$ 

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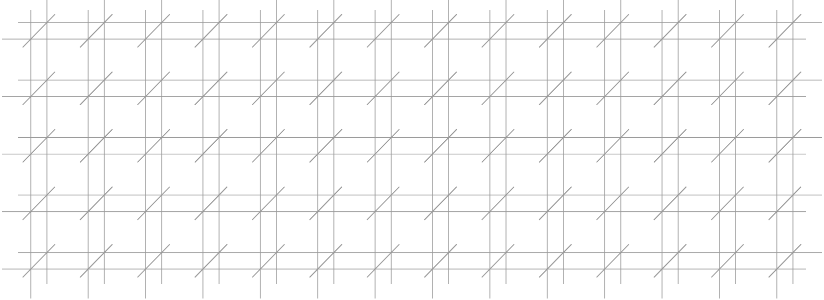
$|\mathrm{GS}\rangle$: all terms $= +1$



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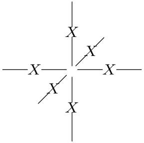
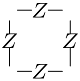
$H = -\sum_{\text{stars}}$ 

$- \sum_{\text{plaquettes}}$ 

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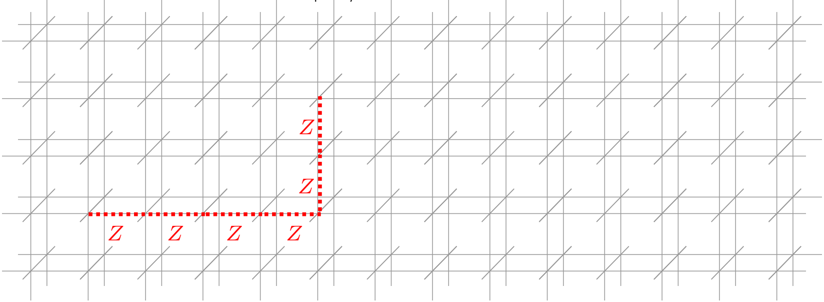
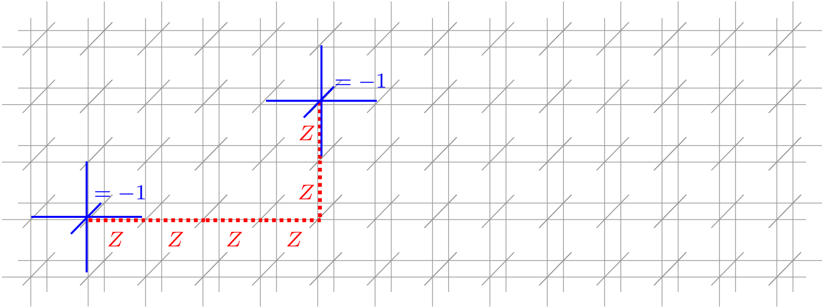


Excitations of the 3D toric code

$$H = -\sum_{\text{stars}} X\,X\,X\,X\,X\,X - \sum_{\text{plaquettes}} Z\,Z\,Z\,Z \tag{5}$$



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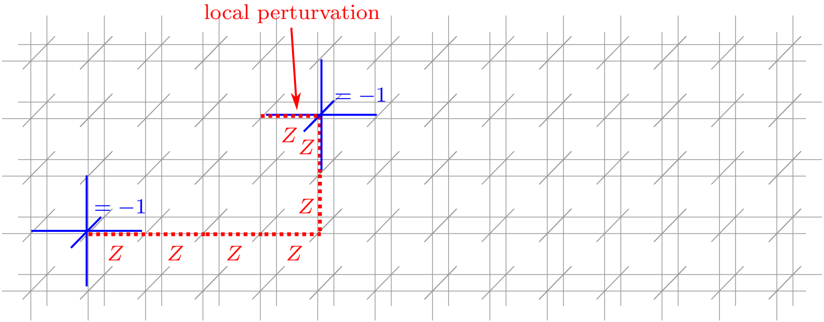

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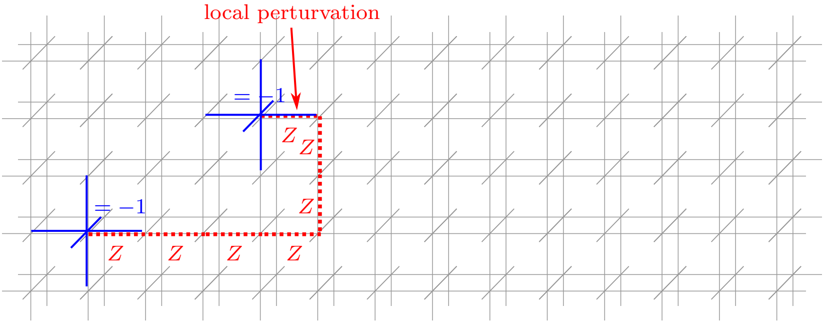

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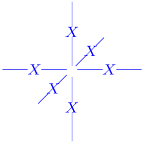
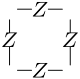
$$H = -\sum_{\text{stars}}$$ 

$$-\sum_{\text{plaquettes}}$$ 

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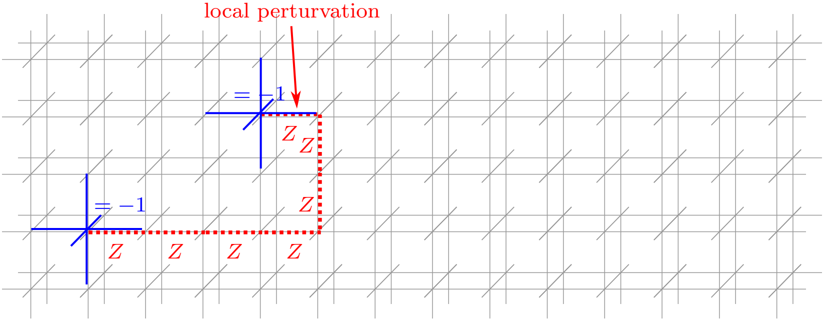

Charge: point-like, mobile

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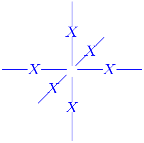
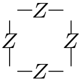
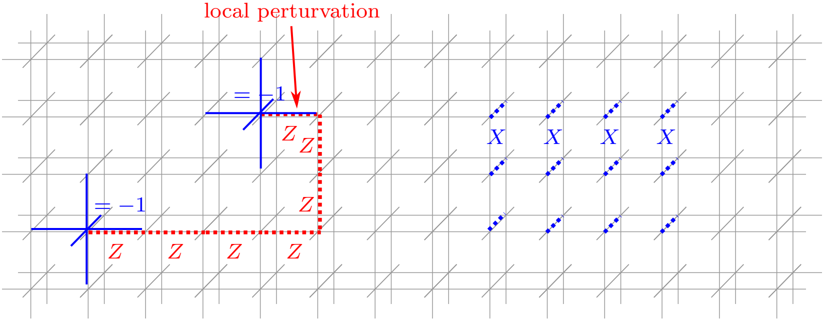
$$H = -\sum_{\text{stars}}$$  $$-\sum_{\text{plaquettes}}$$  (5)



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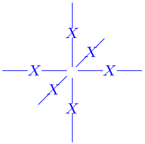
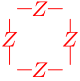
$H = -\sum_{\text{stars}}$ 

$- \sum_{\text{plaquettes}}$ 

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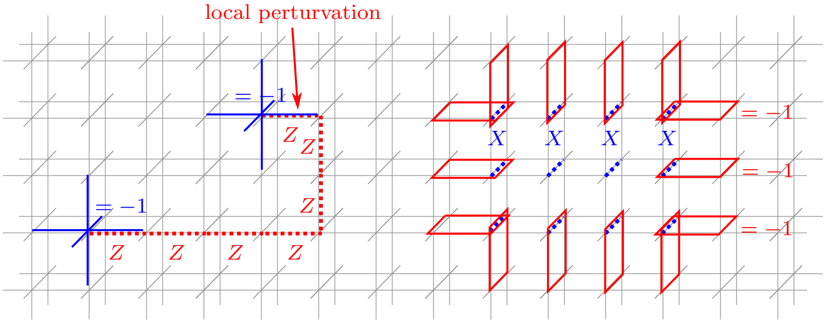

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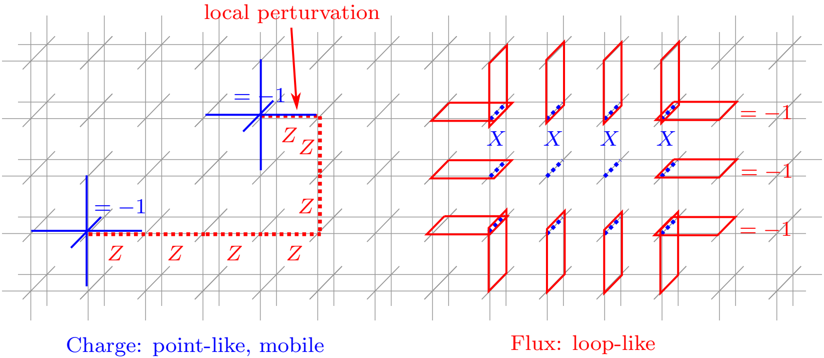
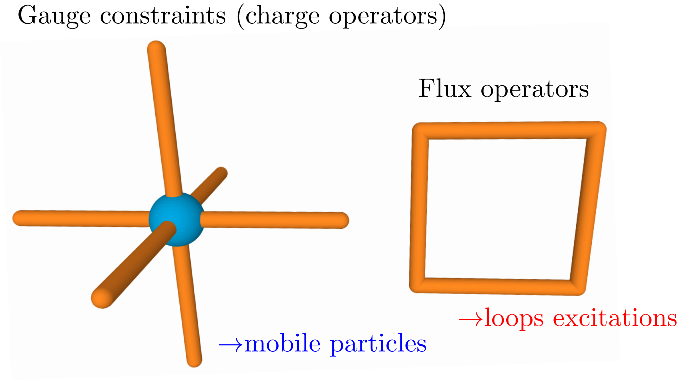


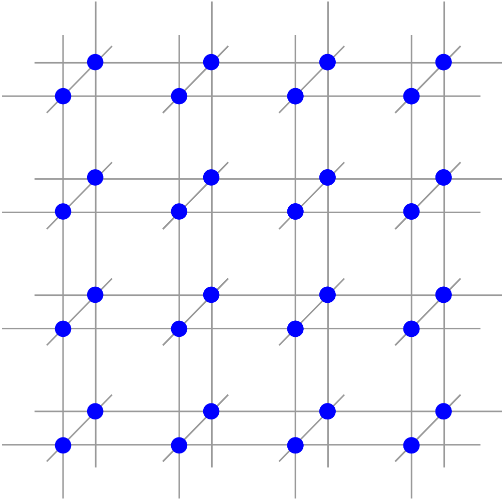
Summary for gauging a global symmetry



Outline

Planar subsystem spin-flip symmetry

System: 3D cubic lattice with one qubit on each site. $G = \left\langle \prod_{\substack{\text{sites} \\ \in\text{plane}}} X \right\rangle := \mathbb{Z}_2^{\text{sub}}$



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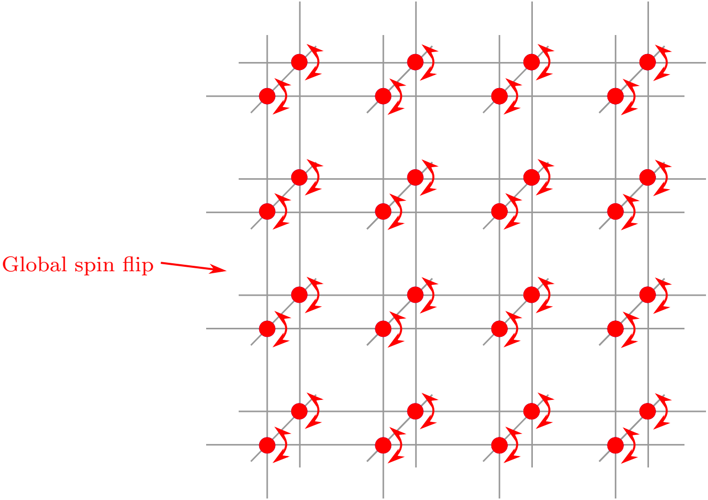

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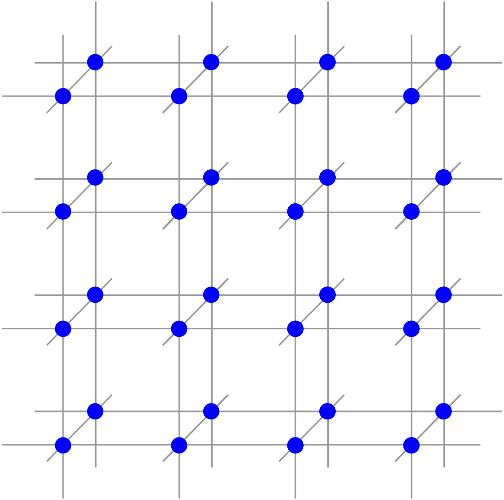

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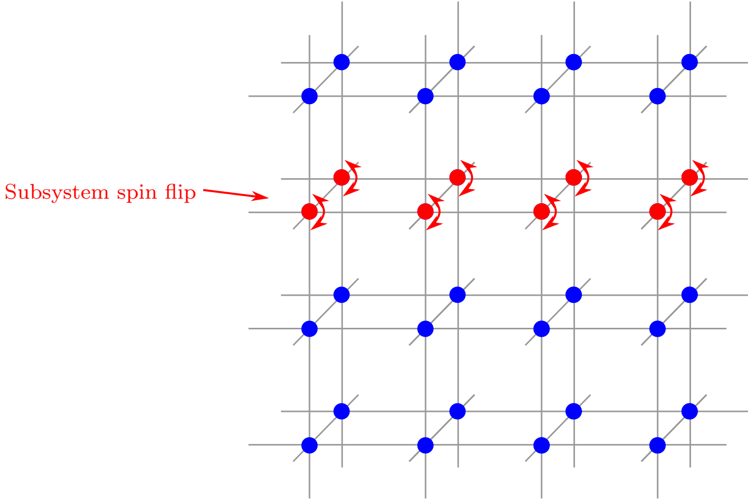

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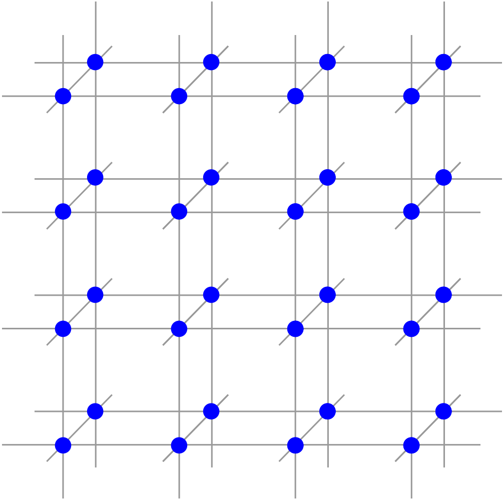

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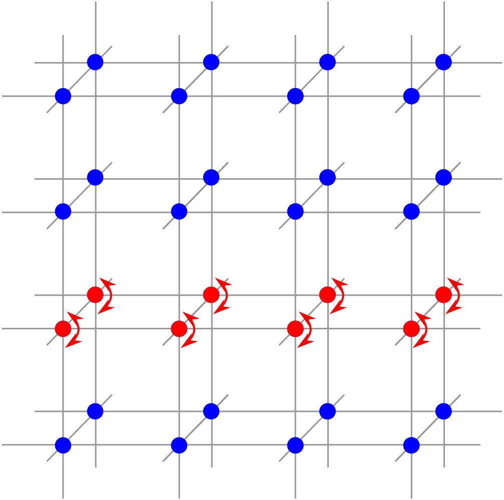

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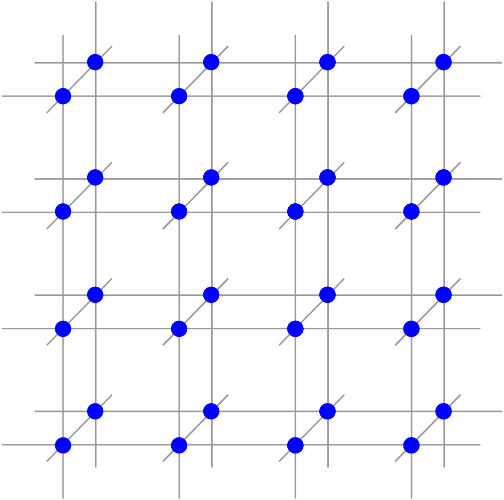

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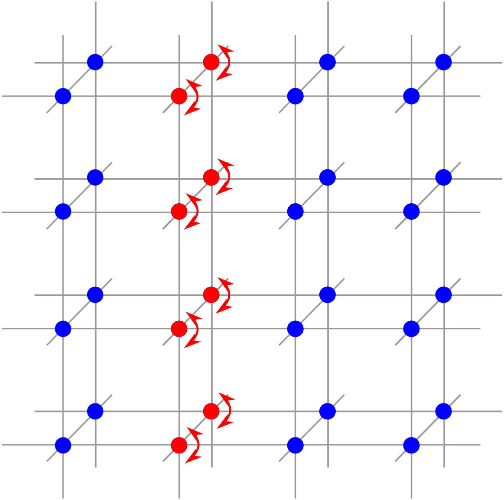

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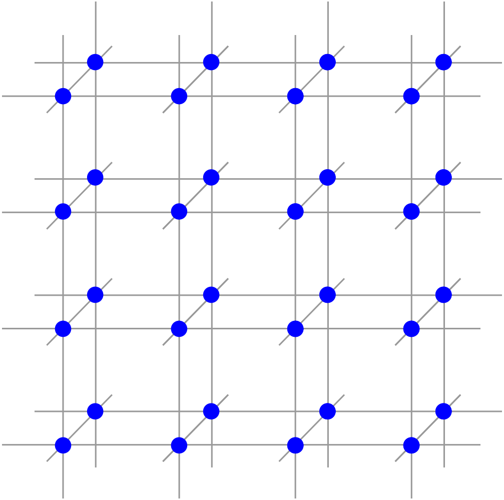

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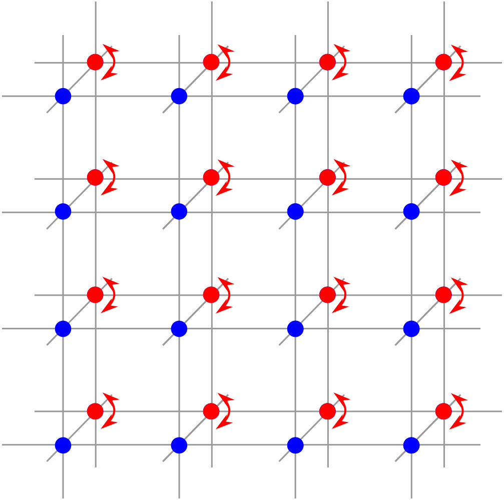

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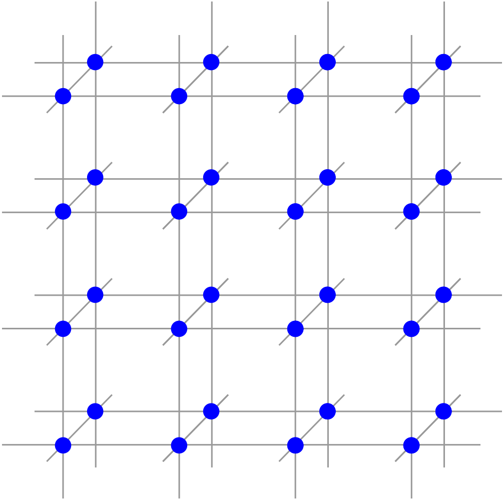

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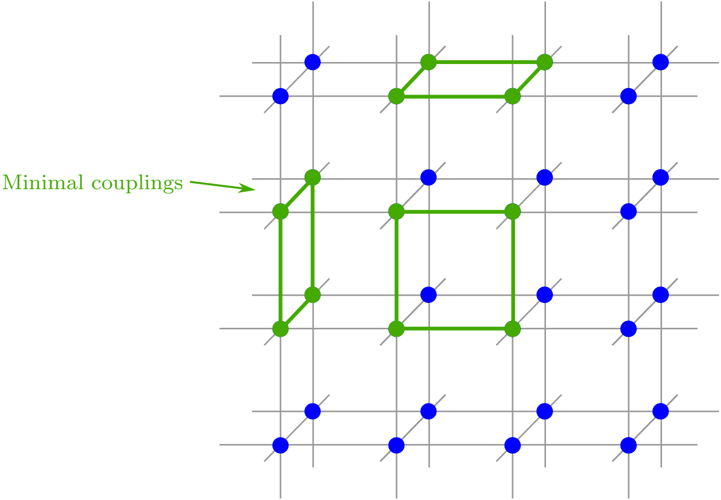

Gauging $\mathbb{Z}_2^{\text{sub}}$ spin-flip symmetry (Shirley, Slagle, Chen)

Starting point: Transverse-field plaquette Ising model on a 3D cubic lattice

$$H = -J \sum_{\text{plaquettes}} \begin{array}{ccc} Z & \text{——} & Z \\ | & & | \\ Z & \text{——} & Z \end{array} - h \sum_{\text{sites}} X \tag{6}$$

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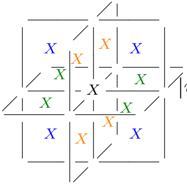
Constraint: 

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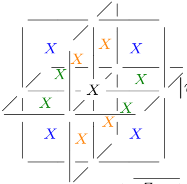
Constraint:  $|\psi\rangle = |\psi\rangle$ (8)

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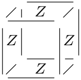
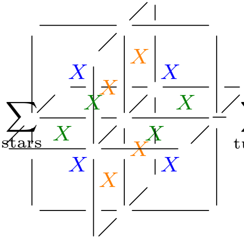
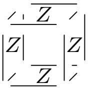
$H_{\text{flux}} = -\sum_{\text{tubes}}$  (9)

Excitations of the X-Cube code

$$H = -\sum_{\text{stars}}$$ 

$$- \sum_{\text{tubes}}$$ 

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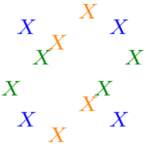
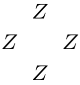
$H = -\sum_{\text{stars}}$  $-\sum_{\text{tubes}}$ 

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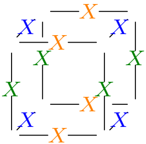
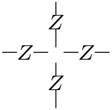
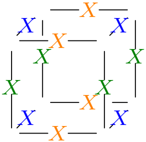
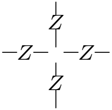
$H = -\sum_{\text{cubes}}$  $-\sum_{\text{crosses}}$  (10)

Excitations of the X-Cube code

$$H = -\sum_{\text{cubes}}$$

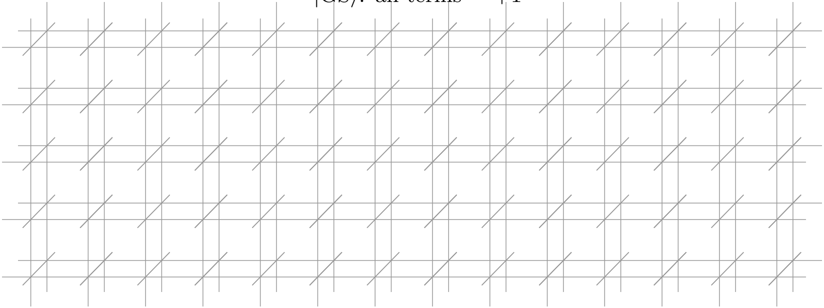


$$-\sum_{\text{crosses}}$$

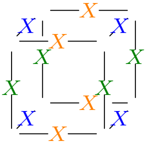
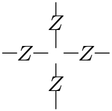


(10)

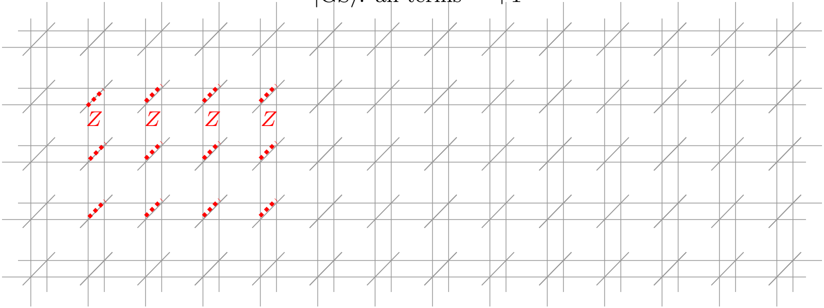
$|\mathrm{GS}\rangle$: all terms $= +1$



Excitations of the X-Cube code

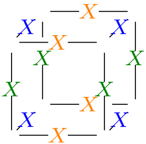
$$H = -\sum_{\text{cubes}}$$  $$-\sum_{\text{crosses}}$$  (10)

$|\mathrm{GS}\rangle$: all terms $= +1$

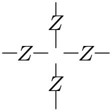


Excitations of the X-Cube code

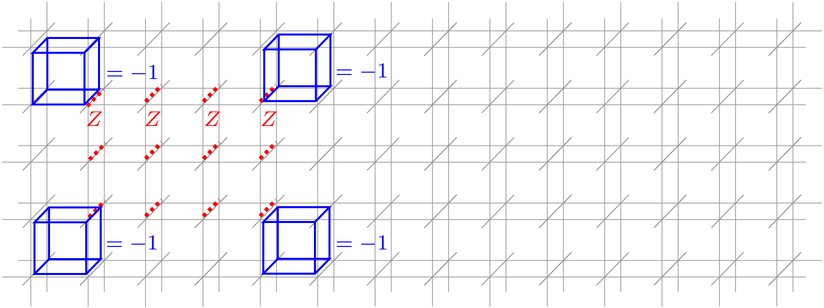
$$H = -\sum_{\text{cubes}}$$



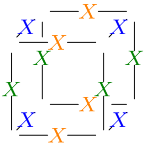
$$-\sum_{\text{crosses}}$$

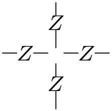


(10)

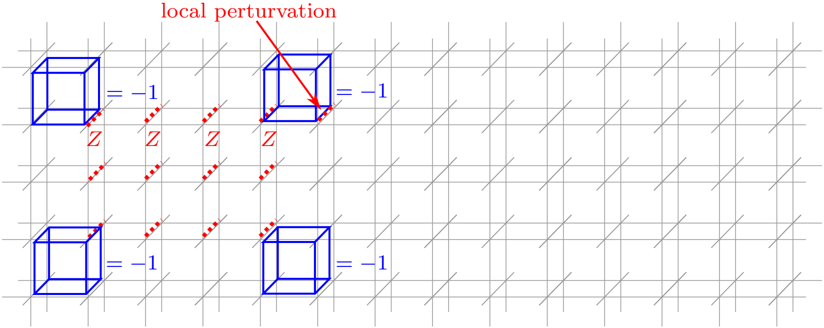


Excitations of the X-Cube code

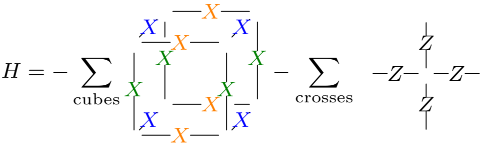
$$H = -\sum_{\text{cubes}}$$ 

$$-\sum_{\text{crosses}}$$ 

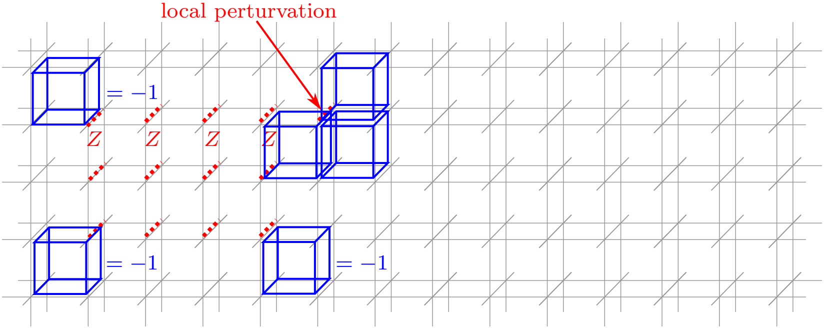
(10)



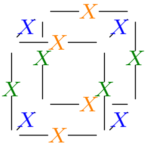
Excitations of the X-Cube code

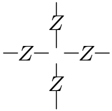


(10)

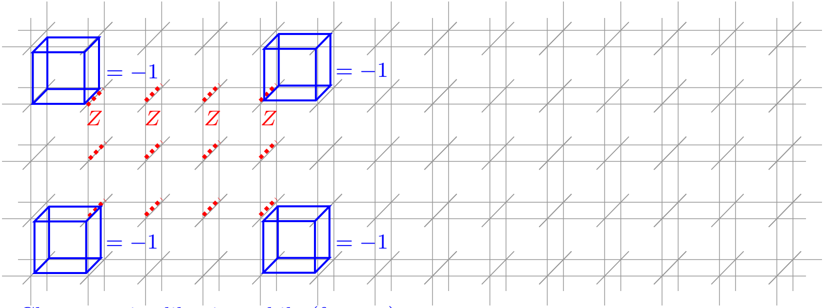


Excitations of the X-Cube code

$$H = -\sum_{\text{cubes}}$$ 

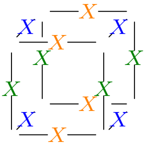
$$-\sum_{\text{crosses}}$$ 

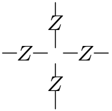
(10)



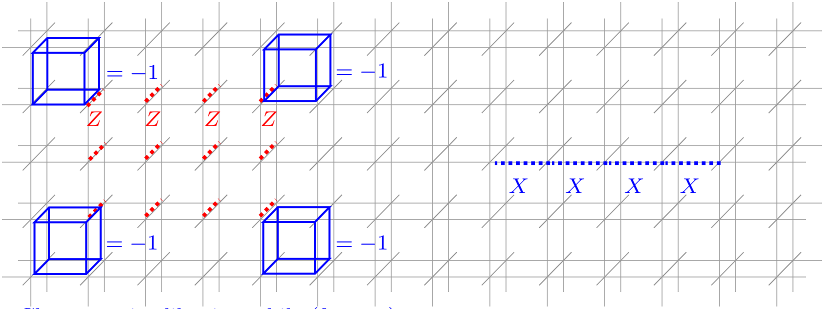
Charge: point-like, immobile (fracton)

Excitations of the X-Cube code

$$H = -\sum_{\text{cubes}}$$


$$-\sum_{\text{crosses}}$$


(10)



Charge: point-like, immobile (fracton)

Excitations of the X-Cube code

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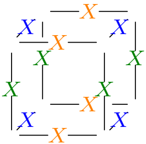
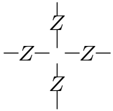
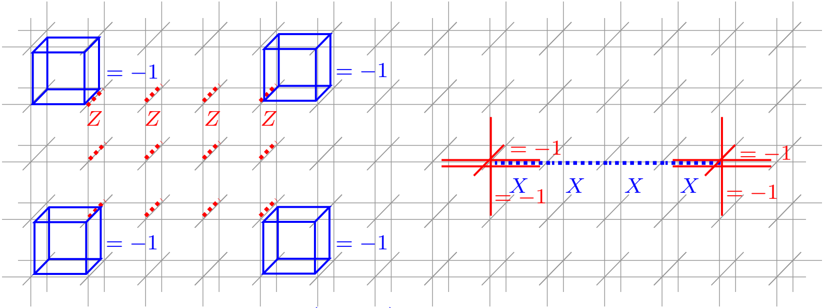


$$-\sum_{\text{crosses}}$$

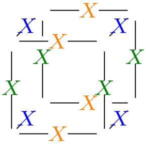


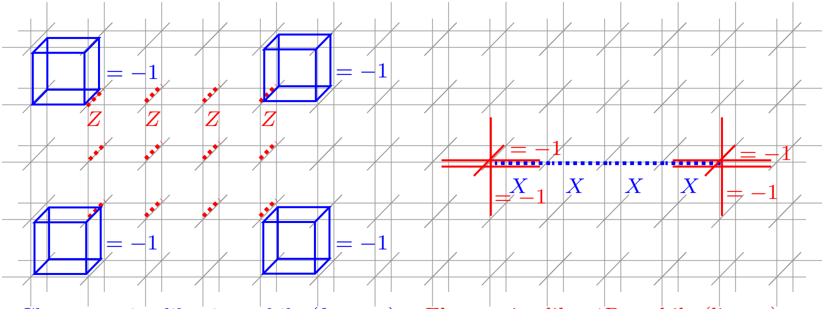
(10)



Charge: point-like, immobile (fracton)

Excitations of the X-Cube code

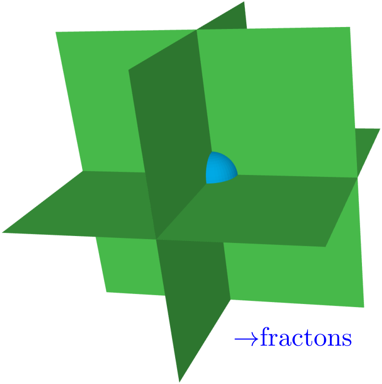
$$H = -\sum_{\text{cubes}}$$  $$- \sum_{\text{crosses}} \begin{array}{ccc} & Z & \\ -Z- & & -Z- \\ & Z & \end{array} \qquad (10)$$



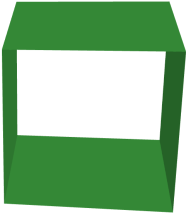
Charge: point-like, immobile (fracton)   Flux: point-like, 1D mobile (lineon)

Summary for gauging a (planar) subsystem symmetry

Gauge constraints (charge operators)



Flux operators

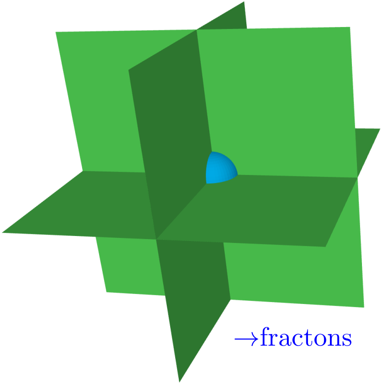


→fractons

→lineons

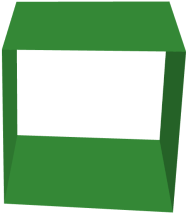
Summary for gauging a (planar) subsystem symmetry

Gauge constraints (charge operators)



→fractons

Flux operators

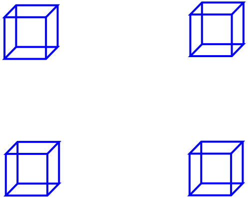


→lineons

Only work for Abelian groups!!

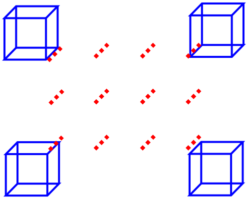
Abelian v.s. non-Abelian fractons

Abelian fractons



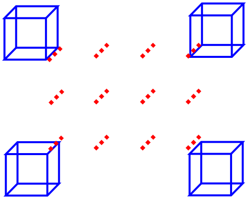
Abelian v.s. non-Abelian fractons

Abelian fractons



Abelian v.s. non-Abelian fractons

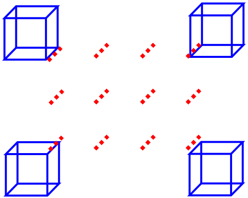
Abelian fractons



- Can always annihilate them

Abelian v.s. non-Abelian fractons

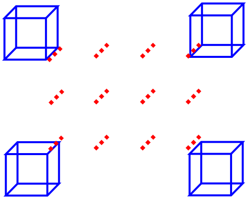
Abelian fractons



- Can always annihilate them
- State is fixed by positions (in obc)

Abelian v.s. non-Abelian fractons

Abelian fractons



- Can always annihilate them
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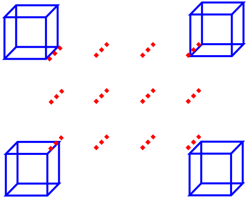
- No non-local information

Abelian v.s. non-Abelian fractons

Abelian fractons



- Can always annihilate them
- State is fixed by positions (in obc)
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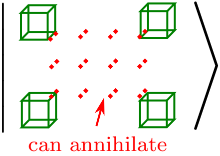
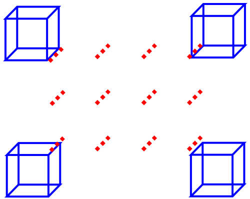
Non-Abelian fractons



Abelian v.s. non-Abelian fractons

Abelian fractons



- Can always annihilate them
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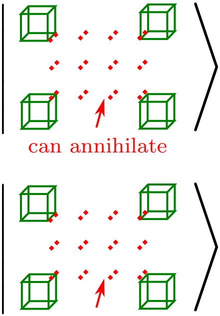
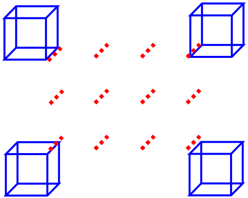
Non-Abelian fractons



Abelian v.s. non-Abelian fractons

Abelian fractons



- Can always annihilate them
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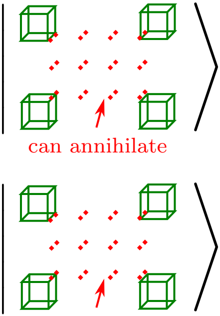
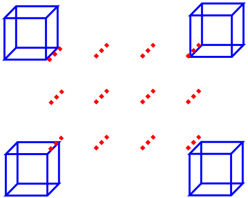
Non-Abelian fractons



- State is not fixed by positions

Abelian v.s. non-Abelian fractons

Abelian fractons



- Can always annihilate them
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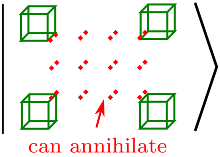
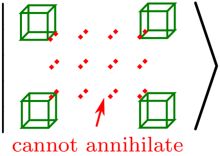
Non-Abelian fractons

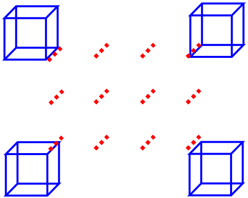




- State is not fixed by positions
- Can store non-local information

Abelian v.s. non-Abelian fractons

Abelian fractons



- Can always annihilate them
- State is fixed by positions (in obc)
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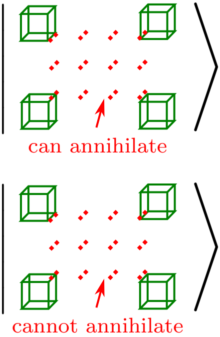
Non-Abelian fractons



- State is not fixed by positions
- Can store non-local information
- Quantum computation?

Outline

Part I: Gauging

Gauging a global symmetry

Gauging a subsystem symmetry

Our construction: gauging a mixture of subsystem and global symmetries

Part II: Algebra of operators

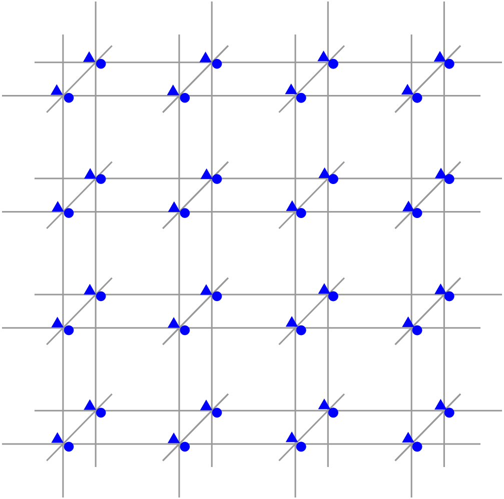
The algebra of Kitaev's quantum double model

The algebra of our fracton model

Gauging $\mathbb{Z}_3^{\text{sub}} \rtimes \mathbb{Z}_2^{\text{glo}}$ symmetry

System: 3D cubic lattice with one qutrit ▲ and one qubit ● on each site.

$$G = \left\langle \prod_{\substack{\text{sites}\\ \in\text{plane}}} XI, \prod_{\substack{\text{sites}\\ \text{(all)}}} SX \right\rangle, \; S := |0\rangle\langle 0| + |1\rangle\langle 2| + |2\rangle\langle 1|$$



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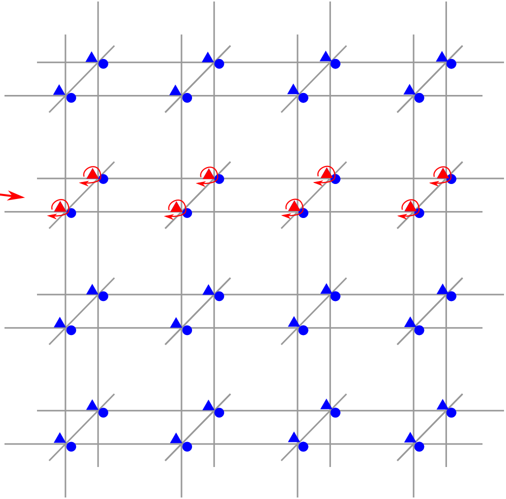
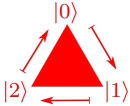
Subsystem qutrit rotation





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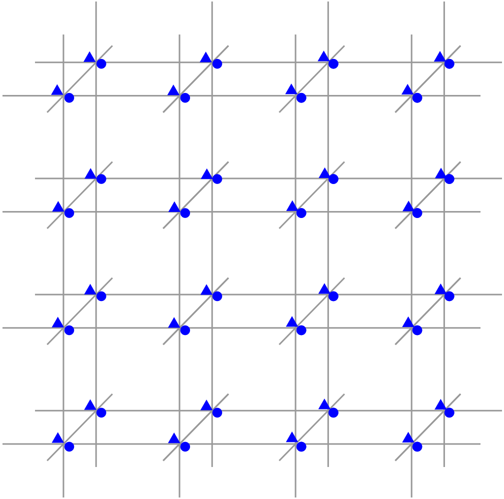

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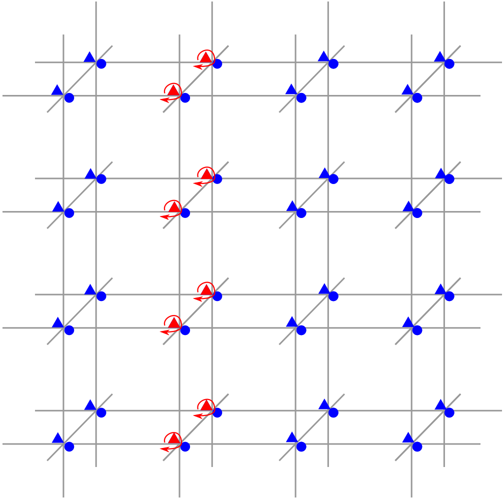

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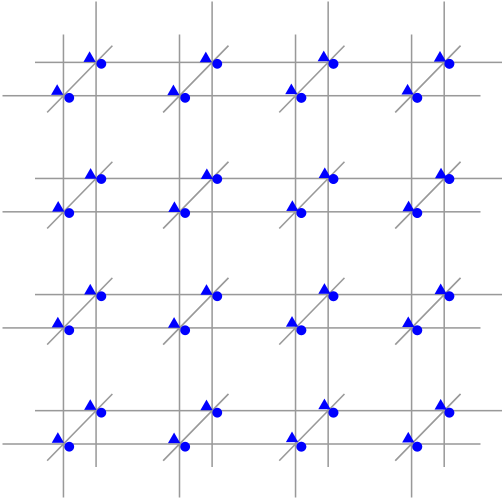

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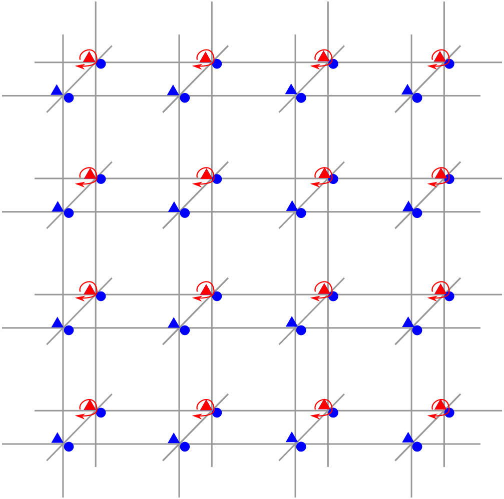

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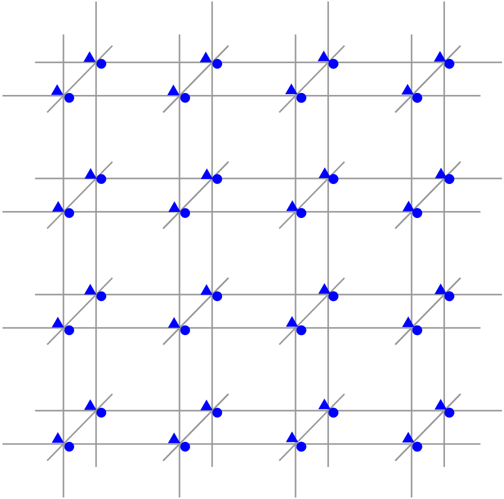

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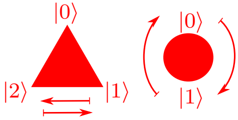
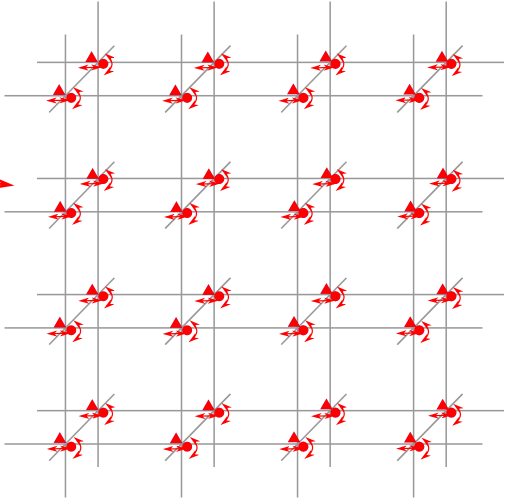
Global qutrit mirroring with qubit flip





Gauging $\mathbb{Z}_3^{\text{sub}} \rtimes \mathbb{Z}_2^{\text{glo}}$ symmetry

System: 3D cubic lattice with one qutrit ▲ and one qubit ● on each site.

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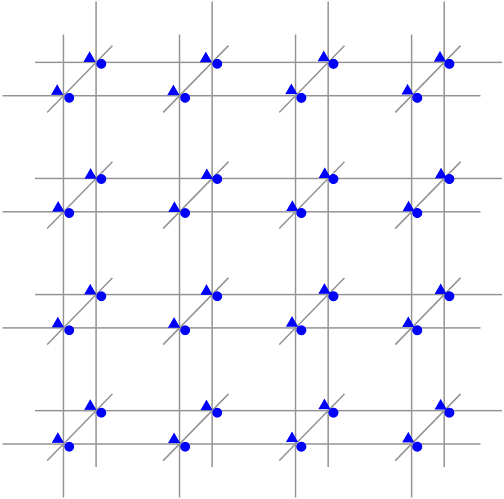

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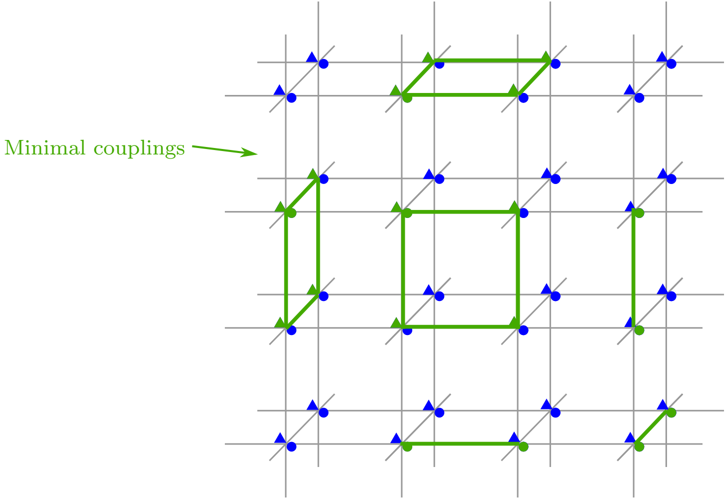

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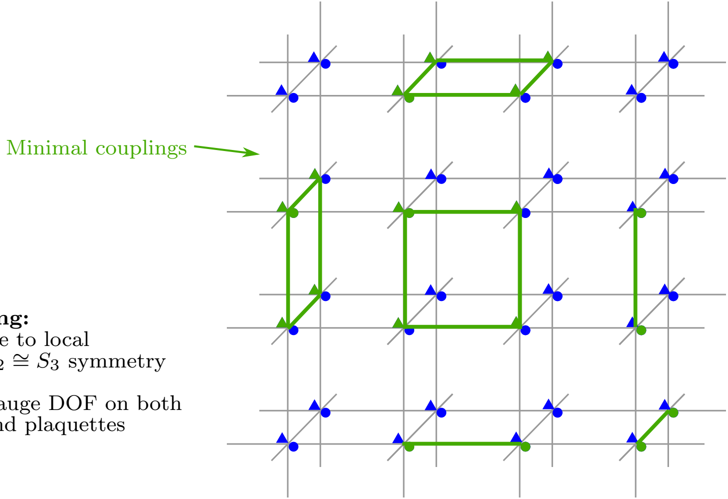
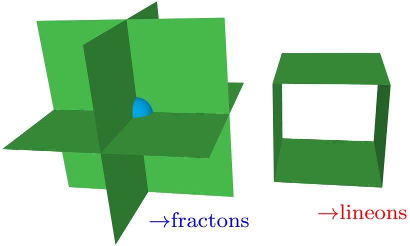


Gauging:
promote to local
$\mathbb{Z}_3 \rtimes \mathbb{Z}_2 \cong S_3$ symmetry

Need gauge DOF on both links and plaquettes

Charges and fluxes

Gauging a subsystem symmetry:



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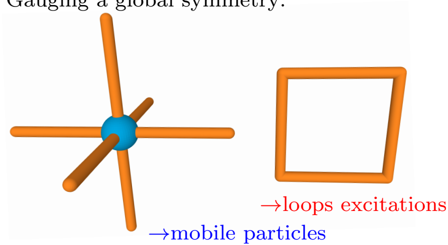

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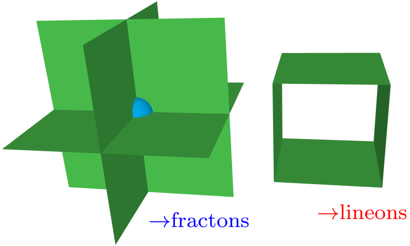

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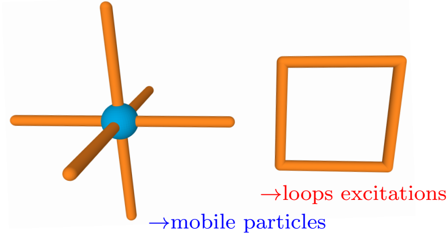
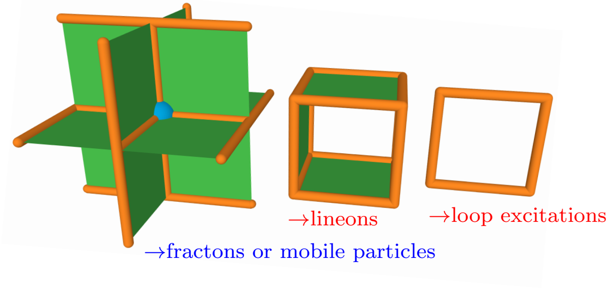


Gauging a mixture of subsystem and global symmetries:



Outline

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Kitaev's quantum double model (QDM)

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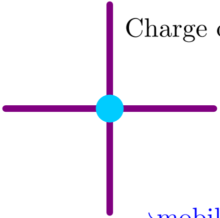
Algebra of QDM

The charge and flux operators satisfy

$$\begin{aligned} &A_{g_1} A_{g_2} = A_{g_1 g_2}, \quad B_{h_1} B_{h_2} = \delta_{h_1, h_2} B_{h_1}, \\ &A_g B_h = B_{g h g^{-1}} A_g, \quad \sum_h B_h = 1. \end{aligned} \tag{13}$$

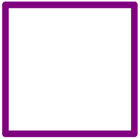
Excitations of QDM



Charge operators

→mobile particles

Flux operators



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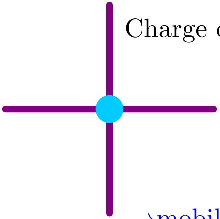

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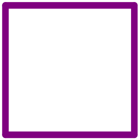
irreducible representations of G

Flux operators



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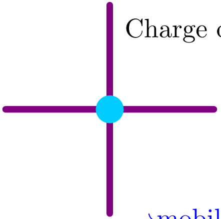

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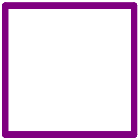

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conjugacy classes of G

Outline

Part I: Gauging

Gauging a global symmetry

Gauging a subsystem symmetry

Our construction: gauging a mixture of subsystem and global symmetries

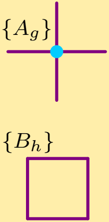
Part II: Algebra of operators

The algebra of Kitaev's quantum double model

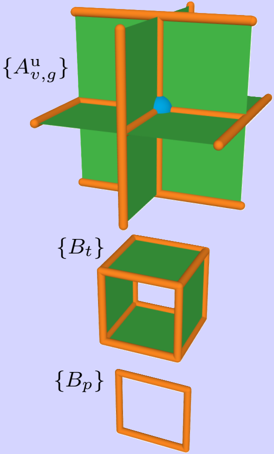
The algebra of our fracton model

The correspondence of the algebras

QDM:



Our fracton model:



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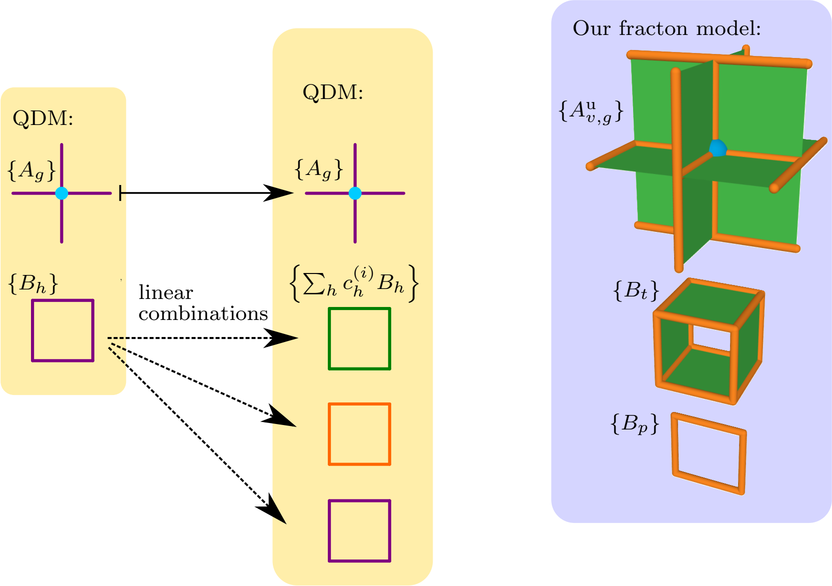

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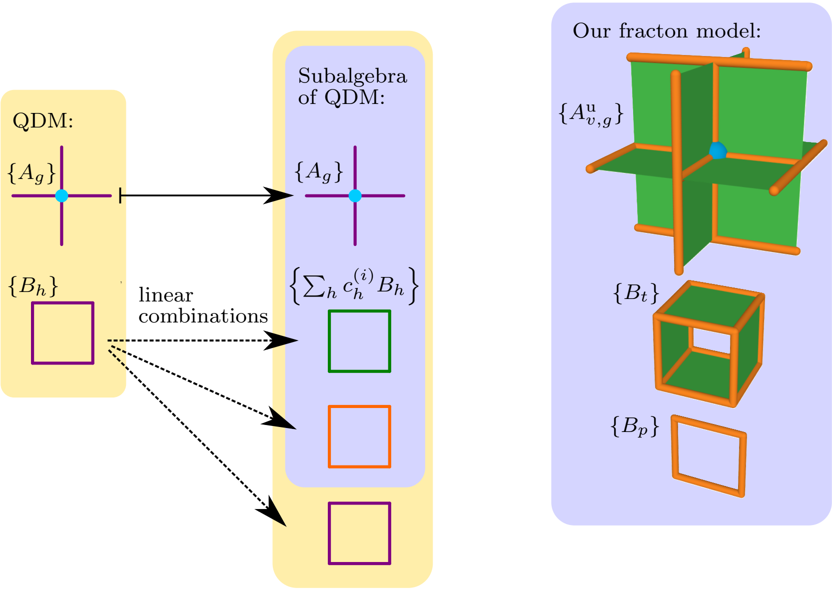

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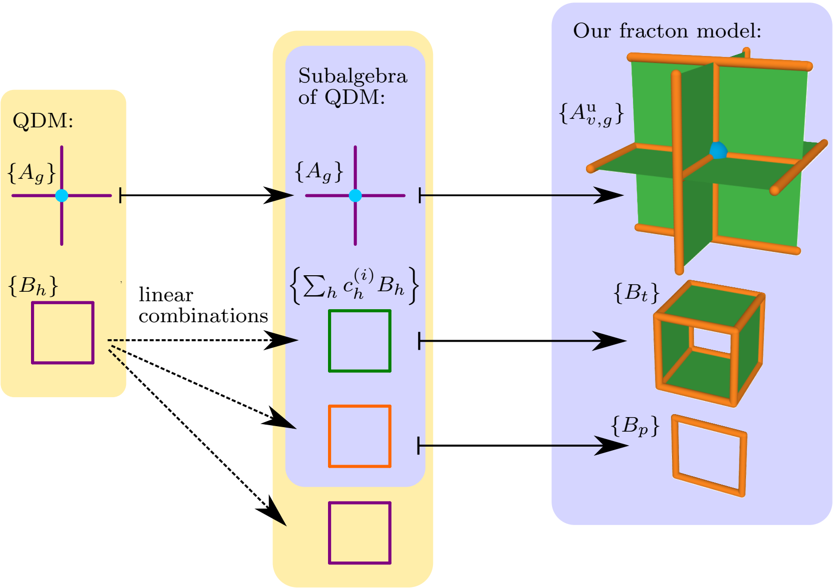

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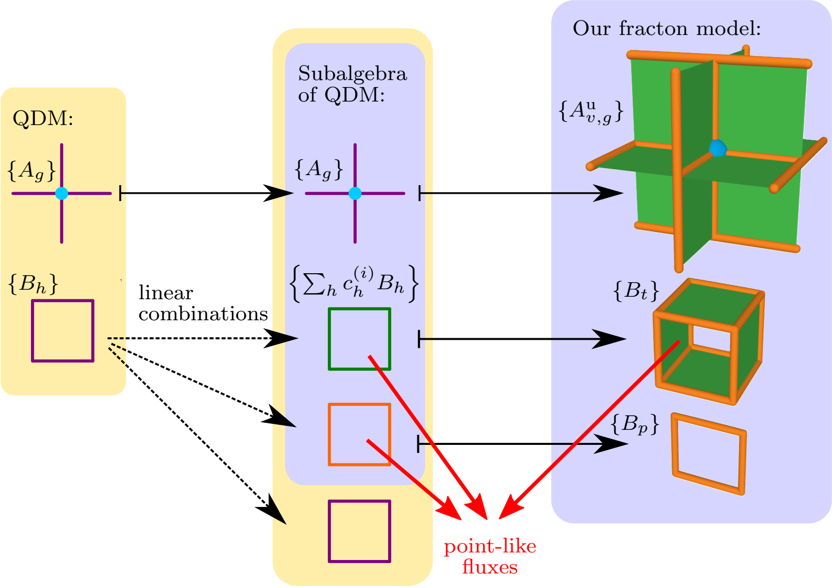

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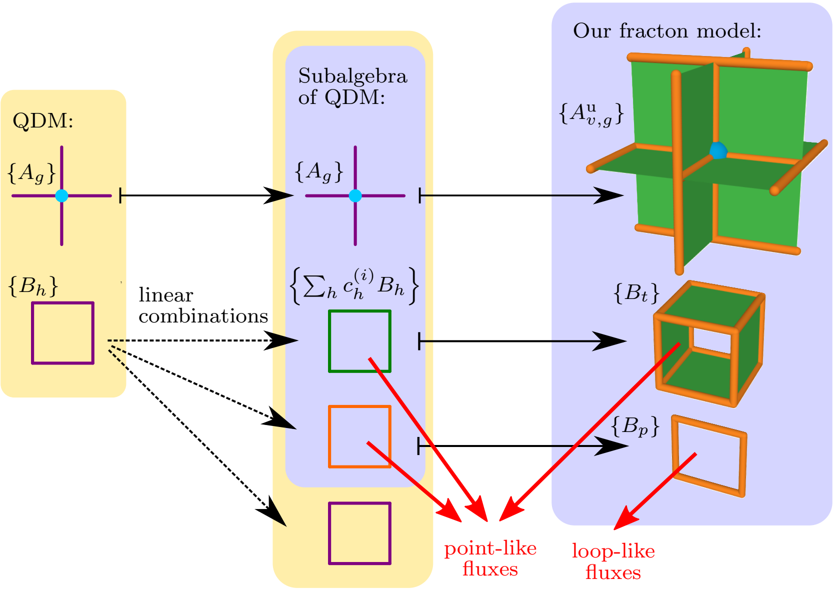

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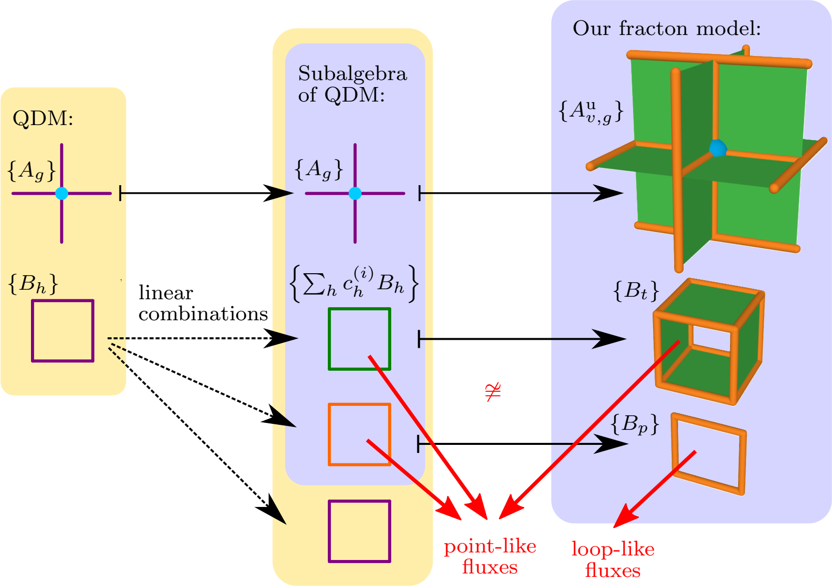

Constraining the Hilbert space

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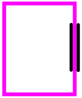
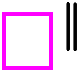
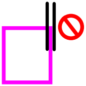
- Loop-like excitations with a thin tube constraint:

Either through the tube



Or not through the tube





≈ Point-like excitation

Classification of excitations

Main Result

With such constraint,

$$\{\text{Charge and flux in gauged } G \text{ fracton model}\} \cong \{\text{Subalgebra of QDM with gauge group } G^{\text{local}}\}, \quad (14)$$

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Example: $G = \mathbb{Z}_3^{\text{sub}} \rtimes \mathbb{Z}_2^{\text{glo}}$, $G^{\text{local}} = S_3$

Charge	Irrep	Type	Flux	Conj class	Type
1	Trivial	Vacuum	1	$\{1\}$	Vacuum
ϕ	Sign	Abelian, mobile	σ	$[g^{(2)}]$	Loop-like
$[f_0]$	2D	Non-Abelian fracton	$[e_d]$	$[g^{(0)}]$	Non-Abelian lineon

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Other examples:

- $G = (\mathbb{Z}_2^{\text{sub}} \times \mathbb{Z}_2^{\text{sub}}) \rtimes \mathbb{Z}_2^{\text{glo}}$, $G^{\text{local}} = D_4$ (Prem, Williamson; Bulmash, Barkeshli)
- $1 \to \mathbb{Z}_2^{\text{sub}} \to G \to \mathbb{Z}_2^{\text{glo}} \to 1$, $G^{\text{local}} = \mathbb{Z}_4$ (Tantivasadakarn, Ji, Vijay)
- $1 \to \mathbb{Z}_2^{\text{sub}} \to G \to K_4^{\text{glo}} \to 1$, $G^{\text{local}} = Q_8$ (Tantivasadakarn, Ji, Vijay)

Summary

1. Gauging a mixture of subsystem and global symmetries can give non-Abelian fracton orders
2. The charge/flux excitations of the gauged model can be classified as in the corresponding quantum double model

See arXiv:2103.08603 for more details.

Future directions:

- Apply to more exotic symmetry (e.g (1D subsystem) $\rtimes$ (2D subsystem) $\rtimes$ global)
- Possible confined-deconfined or Higgsed-deconfined transitions
- Consider other matter (e.g. Majorana fermions)

Gauging $\mathbb{Z}_3^{\text{sub}} \rtimes \mathbb{Z}_2^{\text{glo}}$ symmetry

Starting point: Ising-like matter theory on a 3D cubic lattice (one qutrit and one qubit on each site)

$$H = -J_0 \sum_{\text{plaq.}} \left(\begin{array}{ccc} Z|0\rangle\langle 0| & \text{———} & Z^\dagger I \\ | & & | \\ Z^\dagger I & \text{———} & ZI \end{array} + \begin{array}{ccc} Z^\dagger|1\rangle\langle 1| & \text{———} & ZI \\ | & & | \\ ZI & \text{———} & Z^\dagger I \end{array} \right) - J_2 \sum_{\text{links}} \begin{array}{c} IZ \\ | \\ IZ \end{array}$$

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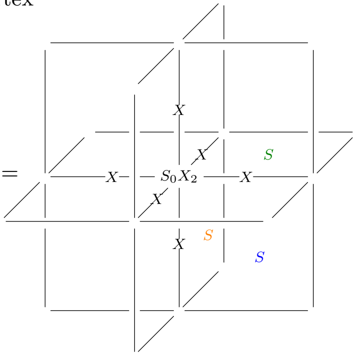
1. Introduce gauge DOF
   - One gauge qutrit on each plaquette.
   - One gauge qubit on each link.

Gauge transformation

2. Enforce gauge symmetry on each vertex

$|\psi\rangle =$  $|\psi\rangle = |\psi\rangle$ (16)

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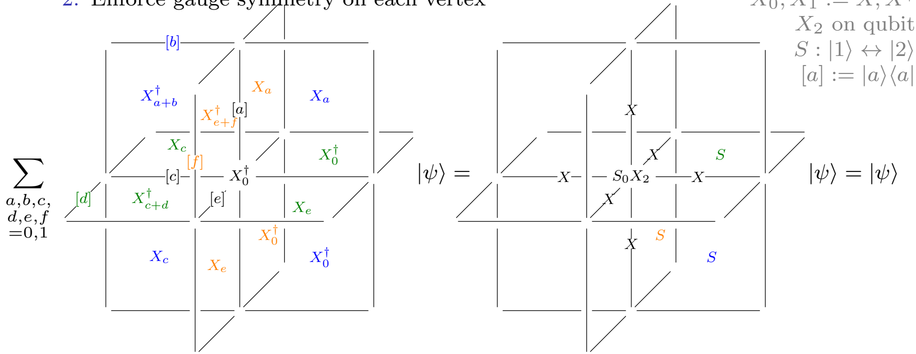
$[a] := |a\rangle\langle a|$



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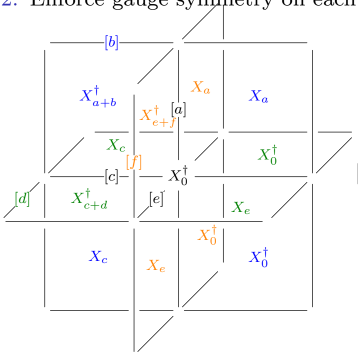
$\sum_{\substack{a,b,c,\\d,e,f\\=0,1}}$



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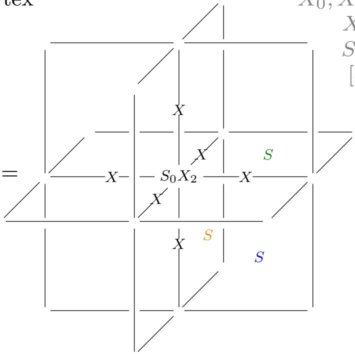
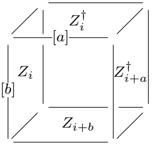


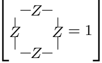
$|\psi\rangle = |\psi\rangle$ (16)

3. Add the gauge flux term

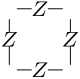
$H_{\text{flux}} = - \sum_{\substack{\text{tubes}\\a,b,i=0,1}}$



$\cdot \Big[$



$\Big] - \sum_{\text{plaquettes}}$



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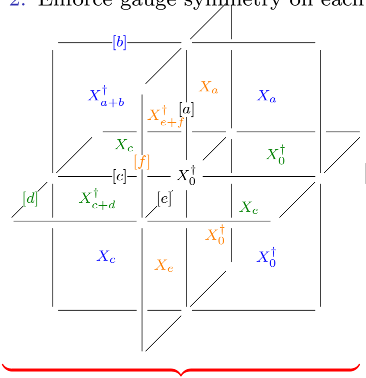
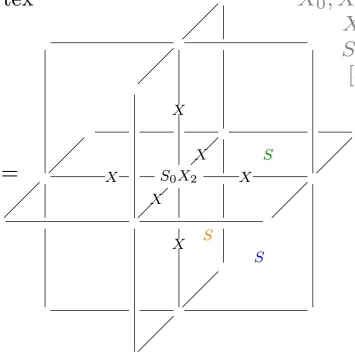
X_2 on qubit
$S : |1\rangle \leftrightarrow |2\rangle$
$[a] := |a\rangle\langle a|$

$$\sum_{\substack{a,b,c,\\ d,e,f\\ =0,1}}$$



$|\psi\rangle =$

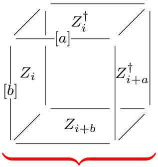


$|\psi\rangle = |\psi\rangle$ (16)

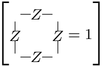
∼ X-cube code

3. Add the gauge flux term

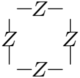
$$H_{\text{flux}} = -\sum_{\substack{\text{tubes}\\ a,b,i=0,1}}$$



$\cdot$



$-\sum_{\text{plaquettes}}$



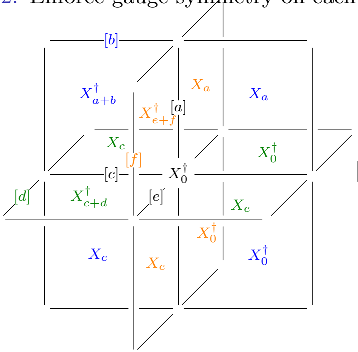
(17)

Gauge transformation

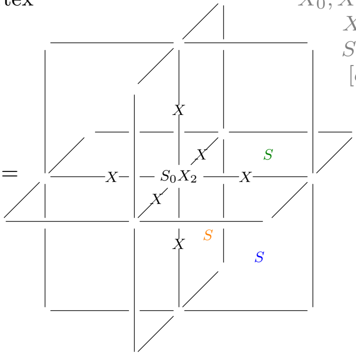
2. Enforce gauge symmetry on each vertex

$X_0, X_1 := X, X^\dagger$
X_2 on qubit
$S : |1\rangle \leftrightarrow |2\rangle$
$[a] := |a\rangle\langle a|$

$\sum_{\substack{a,b,c,\\ d,e,f\\ =0,1}}$



$|\psi\rangle =$

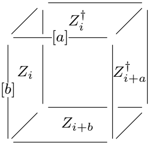


$|\psi\rangle = |\psi\rangle$ (16)

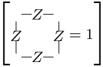
∼ 3D toric code

3. Add the gauge flux term

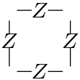
$H_{\text{flux}} = - \sum_{\substack{\text{tubes}\\ a,b,i=0,1}}$



$\cdot$



$- \sum_{\text{plaquettes}}$



(17)

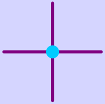
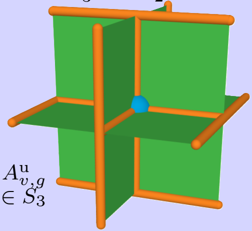
∼ 3D toric code

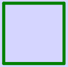
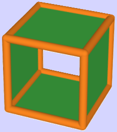
The isomorphism

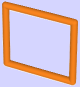
Subalgebra of $\mathbb{Z}_3 \rtimes \mathbb{Z}_2 \cong S_3$ QDM $\cong$ Gauged $\mathbb{Z}_3^{\mathrm{sub}} \rtimes \mathbb{Z}_2^{\mathrm{glo}}$ model:

$g^{(0)} := (1\,2\,3) =$ 

$g^{(2)} := (1\,2) =$ 

 A_g, $g \in S_3$ $\mapsto$ $A^{\mathrm{u}}_{v,g}$, $g \in S_3$ 

 $B_e + e^{\frac{i2\pi}{3}} B_{g^{(0)}} + e^{-\frac{i2\pi}{3}} B_{g^{(0)2}} \mapsto B_t$ 

 $B_e + B_{g^{(0)}} + B_{g^{(0)2}} - B_{g^{(2)}} - B_{g^{(0)}g^{(2)}} - B_{g^{(0)2}g^{(2)}} \mapsto B_p$ 

Algebraic relations:

$B_p^2 = 1$

$B_t^2 = B_t^\dagger$

$B_t^3 = \frac{1}{2}(1 + B_p)$

$B_p B_t = B_t$

$A_g B_p = B_p A_g$

$A_g B_t = B_t A_g$

Gauged Hamiltonian

$$H_{\text{gauged}} = -h \sum_{\text{sites}} (X_0 + X_1 + X_2) - \sum_{p \text{ plaq.}} B_p - \sum_{\substack{t \text{ tube} \\ i=0,1}} B_{t,i} \quad \text{charges and fluxes}$$

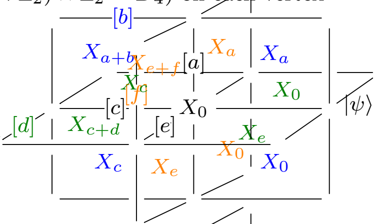
$$-J_0 \sum_{\substack{\text{plaquettes} \\ i,a,b,c,d=0,1 \\ a+b+c+d=0}} J_i \begin{array}{ccc} Z_i[i] & \text{—}[a]\text{—} & Z_{i+a} \\ | & & | \\ [d] & Z_i & [b] \\ | & & | \\ Z_{i+d} & \text{—}[c]\text{-} & Z_{i+a+b} \end{array}$$

$$-J_1 \sum_{\substack{\text{plaquettes} \\ i,a,b,c,d=0,1 \\ a+b+c+d=0}} J_i \begin{array}{ccc} Z_i[i+1] & \text{-}[a]\text{—} & Z_{i+a} \\ | & & | \\ [d] & Z_i & [b] \\ | & & | \\ Z_{i+d} & \text{—}[c]\text{—} & Z_{i+a+b} \end{array} - J_2 \sum_{\text{links}} \begin{array}{c} Z_2 \\ | \\ Z \\ | \\ Z_2 \end{array} \quad \text{gauged couplings}$$

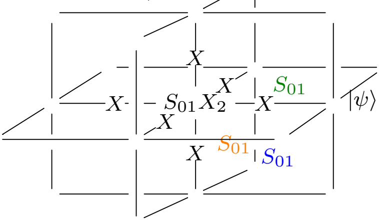
$$-g_0 \sum_{p \text{ plaquettes}} \left(\boxed{X_0} + \boxed{X_1} \right) - g_2 \sum_{\text{links}} \begin{array}{c} | \\ X \\ | \end{array} \quad \text{electric fields}$$

Gauging $(\mathbb{Z}_2^{\mathrm{sub}} \times \mathbb{Z}_2^{\mathrm{sub}}) \rtimes \mathbb{Z}_2^{\mathrm{glo}}$ symmetry

3. Enforce gauge symmetry ($(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \cong D_4$) on each vertex

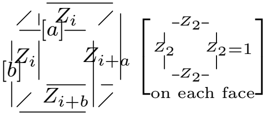
Constraints: $\sum_{a,b,c,d,e,f=0,1}$ 

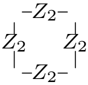
$|\psi\rangle = |\psi\rangle$ (18)



$|\psi\rangle = |\psi\rangle$ (19)

4. Add the gauge flux terms

$H_{\mathrm{flux}} = -\sum_{\substack{\text{tubes}\\ a,b,i=0,1}}$ 

$- \sum_{\text{stars}}$ 

(20)

Bilayer toric code

1. Start with a bilayer toric code

$$\left.\begin{array}{l}\textcolor{blue}{+}\ :\ \text{fracton on the first layer} \\ \textcolor{orange}{+}\ :\ \text{fracton on the second layer}\end{array}\right\}\text{symmetric under layer swap} \tag{21}$$

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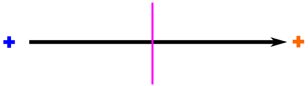

(22)

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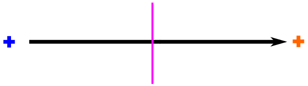

(22)

3. Enforce gauge symmetry

$$\Big(\text{add } \square \text{ and swap inside}\Big)\, |\psi_{\text{physical}}\rangle = |\psi_{\text{physical}}\rangle \tag{23}$$

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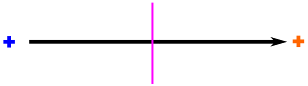

(22)

3. Enforce gauge symmetry

$$\left(\text{add } \square \text{ and swap inside}\right) |\psi_{\text{physical}}\rangle = |\psi_{\text{physical}}\rangle \tag{23}$$

4. Add flux terms so that twist defects cost energy