Positive Partial Transpose criterion in Symplectic geometry

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Background: The Positive Partial Transpose criterion

Quantum Entanglement

Consider a bipartite system consists of subsystems A and B.

Definition

A bipartite state $\hat{\rho}$ is called **separable** if it can be written as

$$\hat{\rho} = \sum_i p_i \hat{\rho}_{A,i} \otimes \hat{\rho}_{B,i} \tag{1}$$

for some sets of density operators $\{\hat{\rho}_{A,i}\}$ and $\{\hat{\rho}_{B,i}\}$ and positive real numbers $\{p_i\}$ satisfying $\sum_i p_i = 1$.

Otherwise, $\hat{\rho}$ is called **entangled**.

e.g. $\frac{|00\rangle + |11\rangle}{\sqrt{2}} \frac{\langle 00| + \langle 11|}{\sqrt{2}}$ is entangled, but $\frac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2}$ is separable.

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Given $\hat{\rho}$, how to test if it is entangled?

Positive Partial Transpose criterion[1]

$$\hat{\rho} = \sum_{ijkl} \rho_{ij,kl} (|i\rangle_A \otimes |j\rangle_B)({}_A\langle k| \otimes {}_B\langle l|)$$

[1] A. Peres, Phys. Rev. Lett. 77, 1413–1415 (1996).

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Theorem

If $\hat{\rho}$ is separable, then

$$\hat{\rho}^{T_B} \geq 0\,. \tag{2}$$

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If $\hat{\rho}^{T_B}$ is not a valid state, then $\hat{\rho}$ is entangled.

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Motivation: PPT in Continuous variable systems

The continuous variable quantum phase space

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Observables:

$\hat{q}_1, \cdots \hat{q}_N, \hat{p}_1, \cdots \hat{p}_N$

Relations:

$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$

$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$

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Observables:

$\hat{\boldsymbol{\xi}} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)^T$

Relations:

$[\hat{\xi}_k, \hat{\xi}_l] = i\Omega_{kl}$

$$\boldsymbol{\Omega} = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}$$

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Wave function:

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→ Wigner-Weyl transform →

Wigner function:

$W(\boldsymbol{\xi}) = W(q_1, p_1, \cdots, q_N, p_N)$

Examples of Wigner function

- Coherent states
  `coherent.gif`
  (image source:
  `https://commons.wikimedia.org/wiki/File:SmallDisplacedGaussianWF.gif`)
- Fock states
  `fock.jpg`
  (image source: `https://en.wikipedia.org/wiki/File:Wigner_functions.jpg`)

PPT criterion for Continuous variable systems[2]

PPT: If a state is separable,
then its **partial transpose is a valid state.**

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$$W_{\hat{\rho}^{T_B}}(q_A, p_A, q_B, p_B) = W_{\hat{\rho}}(q_A, p_A, q_B, -p_B) \tag{3}$$

in both the position and Fock basis.

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Theorem

If $W(\boldsymbol{\xi})$ is a Gaussian function, then it is a valid state iff

$$\boldsymbol{\sigma} + i\boldsymbol{\Omega} \geq 0\,, \tag{4}$$

where $\sigma_{ij} = \langle \xi_i \xi_j + \xi_j \xi_i \rangle - 2\langle \xi_i \rangle \langle \xi_j \rangle$.

It is a form of the uncertainty principle.

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Theorem

A bipartite Gaussian function $W(\boldsymbol{\xi})$ **is a valid state** iff

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A change from $\boldsymbol{\Omega}_A \oplus \boldsymbol{\Omega}_B$ to $\boldsymbol{\Omega}_A \oplus -\boldsymbol{\Omega}_B$.

Symplectic structure of Phase spaces

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$\boldsymbol{\Omega}$ is actually a matrix representation of a bilinear form ω:

$$\omega(\xi_i, \xi_j) = \Omega_{ij} \tag{7}$$

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Definition

A **symplectic vector space** (E, ω) is a vector space E together with a nondegenerate, antisymmetric bilinear form ω, called the **symplectic form**.

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Can we formulate PPT with geometry?

Result: PPT criterion in Symplectic geometry (CV systems)

Partial transpose in symplectic geometry

Phase space of A:
(E_A, ω_A)

Phase space of B:
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Partial transpose in symplectic geometry

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Phase space of B:
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Phase space of the whole system:
$(E_A \oplus E_B, \omega_A + \omega_B) = (E, \omega)$

Let A, B has n, m modes respectively, then the whole system has $N = n + m$ modes and $E_A \cong \mathbb{R}^{2n}$, $E_B \cong \mathbb{R}^{2m}$, $E \cong \mathbb{R}^{2N}$.

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Proposition

The partial transposition map (with respect to a quadrature basis) corresponds to a separable linear symplectomorphism from (E, ω) to (E, ω_-).

Moreover, any such symplectomorphisms correspond to the partial transposition map up to a local unitary transformation.

Wigner functions
of Gaussian states

Positivity in symplectic geometry

Wigner functions of Gaussian states

All $2N$-dimensional Gaussian distributions

Positivity in symplectic geometry

Wigner functions of Gaussian states

All $2N$-dimensional Gaussian distributions

$\longleftrightarrow$

$\{(\xi, \sigma)\}$

ξ: expectation vector

σ: covariance form

Positivity in symplectic geometry

Wigner functions of Gaussian states ⟷ $\{(\xi, \sigma) \mid \sigma + i\omega \geq 0\}$

All $2N$-dimensional Gaussian distributions ⟷ $\{(\xi, \sigma)\}$

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Generalization:

Wigner functions of certain type of states

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Larger set of functions invariant under partial transposition

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v_i: geometric objects

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All $2N$-dimensional Gaussian distributions ⟷ $\{(\xi, \sigma)\}$

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Generalization:

Wigner functions of certain type of states ⟷ $\{(v_1, \cdots, v_k) \mid P(v_1, \cdots, v_k; \omega)\}$

P: geometric condition

Larger set of functions invariant under partial transposition ⟷ $\{(v_1, \cdots, v_k)\}$

v_i: geometric objects

PPT in symplectic geometry[3]

Summary of our framework:

1. Extend the set of Wigner functions to a larger set invariant under partial transposition.
2. Parametrize this set with geometric objects $v_1, \cdots, v_k$.
3. Find a geometric condition $P(v_1, \cdots, v_k; \omega)$ that the parameters correspond to a valid state.

Definition

An element $(v_1, \cdots, v_k)$ of the above parametrization is called **valid with respect to a symplectic form** ω' if the condition $P(v_1, \cdots, v_k; \omega')$ is true.

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Proposition

A state satisfies PPT iff it is valid with respect to ω_-.

[3] Yi-Ting Tu and Ray-Kuang Lee, in preparation

Example: Bipartite cat states

A cat state is a superposition of two coherent states at opposite positions.

$$|\text{cat}\rangle \propto |\alpha\rangle \pm |-\alpha\rangle\,, \tag{8}$$

`cat-ani.gif`
(image source: `https://commons.wikimedia.org/wiki/File:Wigner_function_of_a_Schr%C3%B6dinger_cat_state.gif`)

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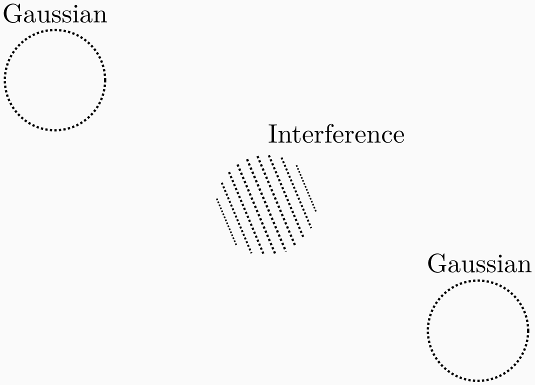
We consider the following type of bipartite cat states:

$$|\text{cat}_{\boldsymbol{\xi}}\rangle \propto |\boldsymbol{\xi}\rangle - |-\boldsymbol{\xi}\rangle\,, \tag{9}$$

where $\boldsymbol{\xi}$ is a vector in the phase space $E \cong \mathbb{R}^{2N}$.

Example: Bipartite cat states



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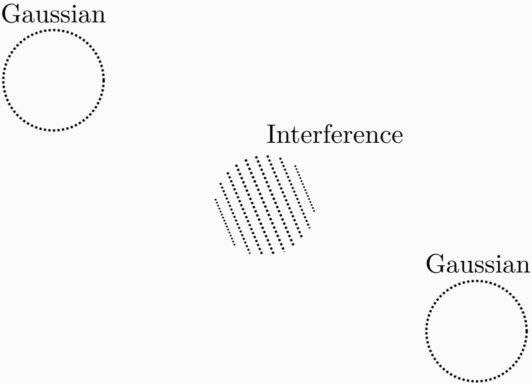

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Same functions but interference may be in wrong direction

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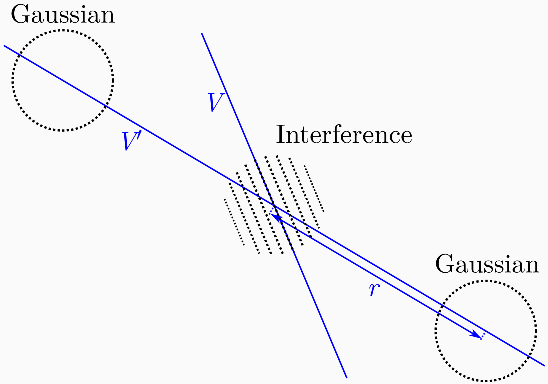

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⟷

$\{(V, V', r)\}$
V: a $(2N-1)$-dim subsp,
V': a 1-dim subsp, r: a scalar

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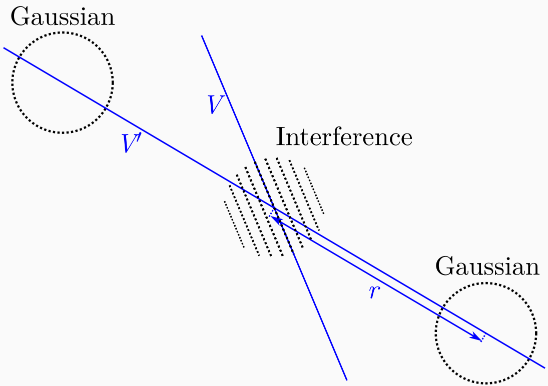

Wigner functions of bipartite cat states ⟷ $\{(V, V', r) \mid V' = V^{\omega}\}$, where $V^{\omega} := \{u \in E \mid \omega(u, v) = 0 \, \forall v \in V\}$

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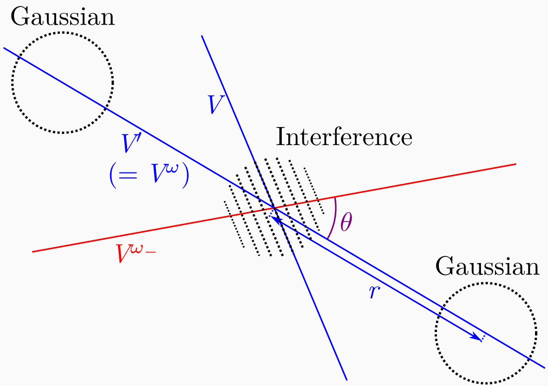

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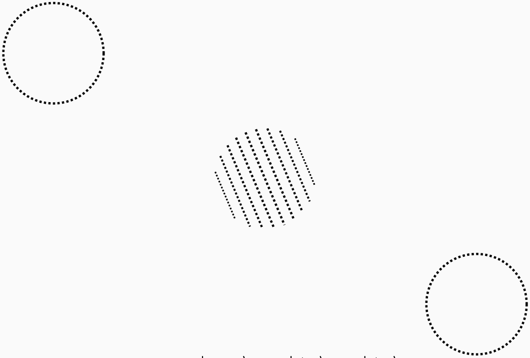

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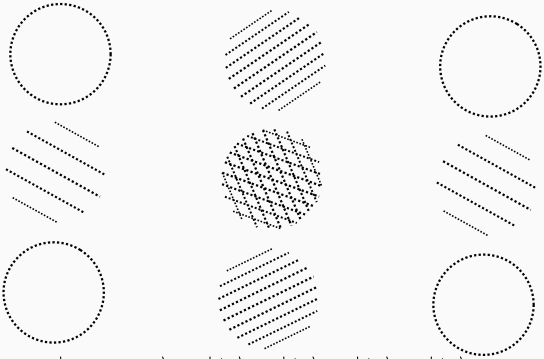

$$|\text{compass}\rangle \propto |\xi_1\rangle + |\xi_2\rangle + |\xi_3\rangle + |\xi_4\rangle$$

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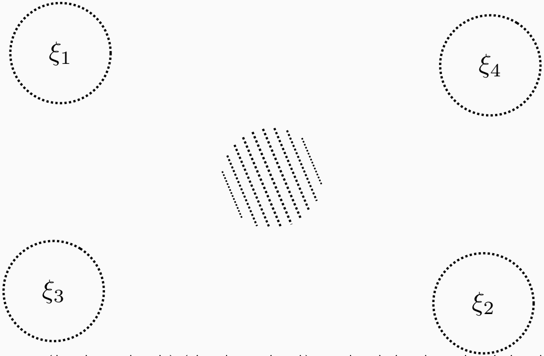

$$\rho \propto (|\xi_1\rangle + |\xi_2\rangle)(\langle\xi_1| + \langle\xi_2|) + |\xi_3\rangle\langle\xi_3| + |\xi_4\rangle\langle\xi_4|$$

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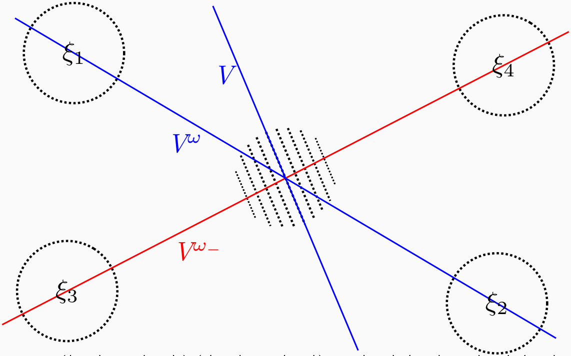

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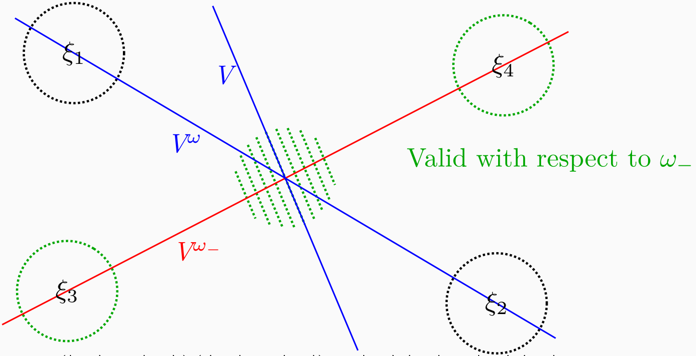

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Example: Zero eigenstates of quadratures

Consider the simultaneous zero eigenstates of N compatible quadrature operators.

(For example, the $p_1 + p_2 = 0, x_1 - x_2 = 0$ eigenstate.)

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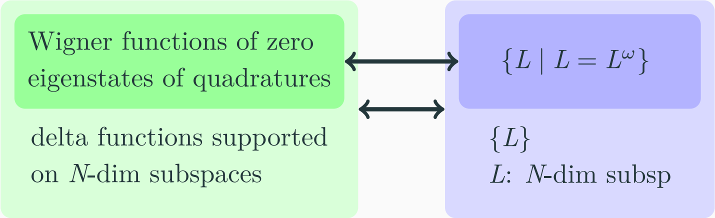
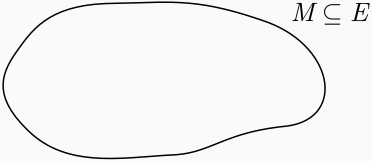


Relationship with classical integrable system

Let M be the Lagrangian torus of a bipartite classical system.



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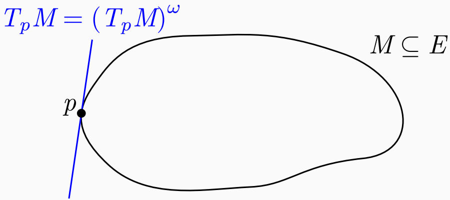

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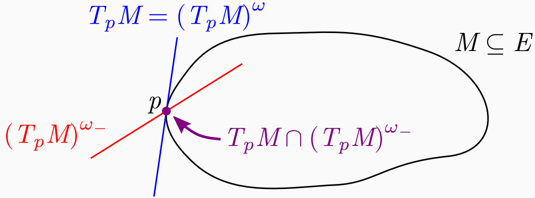

Remark: Testing PPT with symplectic complements

In the examples above, the condition of validity $P(v_1, \cdots, v_k; \omega)$ involves only the operation of taking the symplectic complement: $P(V, V', \cdots) \iff V' = V^{\omega}$.

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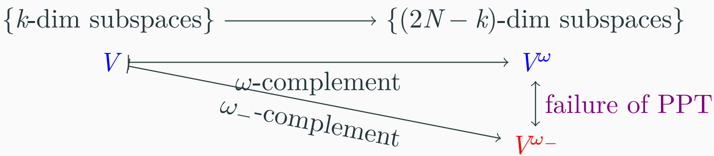
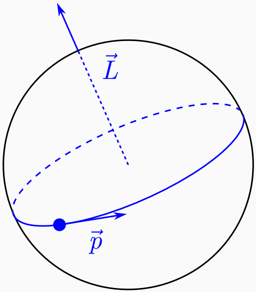


Result: PPT criterion in Symplectic geometry (general systems)

From CV to spin – The phase space

Consider a classical system of a free particle on a sphere.



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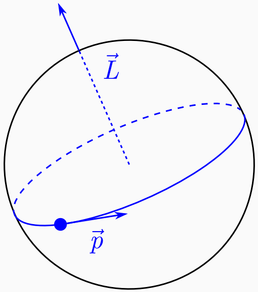

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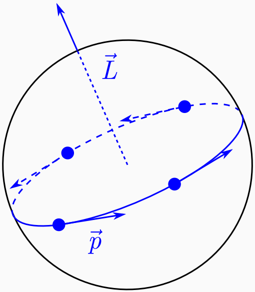

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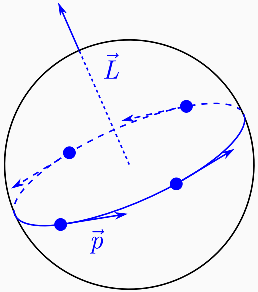


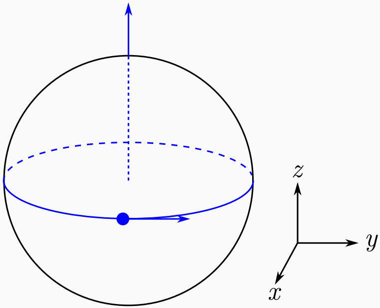
The phase space is 4-dimensional. Do the following reduction:

- Restrict $|\vec{L}|$ to a fixed value L.
- Identify all states related by time evolution as a single state.

Then the reduced phase space is 2-dimensional and identified with the sphere of all possible $\vec{L}$'s. (Large spin limit of spin phase spaces.)

From CV to spin – The symplectic structure

Consider two perturbations of this classical state:



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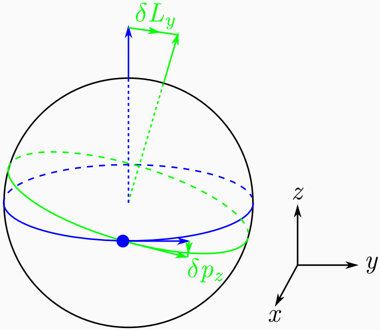

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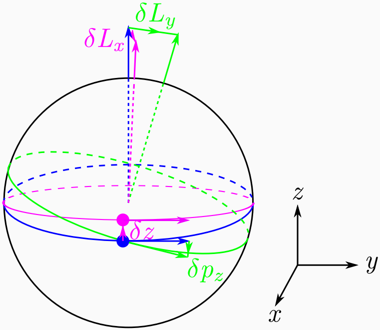

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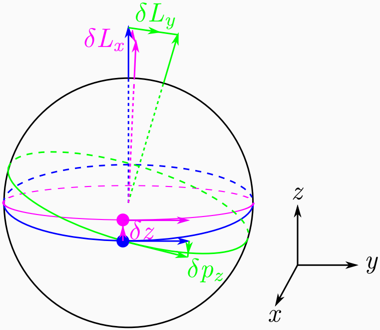

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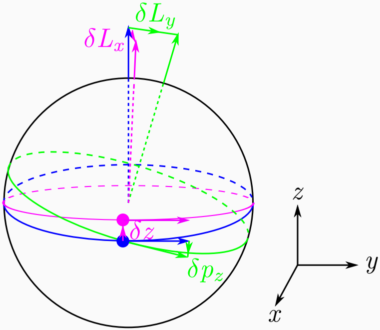

From this we can see that:

- The symplectic form is reduced as $\omega(\cdots) = \delta z \delta p_z = -\frac{1}{L}\delta L_x \delta L_y$.
- The reduced phase space locally looks like that of a one-dimensional particle with $q \propto \delta L_x, p \propto \delta L_y$.

SW correspondence[4] and the KKS symplectic form[5]

We use the framework of **Stratonovich-Weyl correspondence**:

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We assume that the phase space can be identified with a coadjoint orbit of G, so its symplectic form can be given by the **Kirillov-Kostant-Souriau form**.

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SW correspondence and the KKS symplectic form

	CV systems		
Hilbert space H	$L^2(\mathbb{R}^N)$		
Dynamical group G	Heisenberg group H_{2N+1}		
Reference state $\|\psi_0\rangle$	SHO ground state $\|0\rangle$		
Phase space X	$\mathbb{R}^{2N}$		
SW kernel $\hat{\Delta}(\xi)$	$\int \frac{d^{2N}\boldsymbol{\xi}'}{\pi^2} \cdot e^{i\boldsymbol{\xi}'^T \boldsymbol{\Omega} \boldsymbol{\xi}} \hat{D}(\boldsymbol{\xi})$		
Symplectic form ω	$\sum_i dq_i \wedge dp_i$		

SW correspondence and the KKS symplectic form

	CV systems	Spin systems	
Hilbert space H	$L^2(\mathbb{R}^N)$	$\mathbb{C}^{2j+1}$	
Dynamical group G	Heisenberg group H_{2N+1}	SU(2)	
Reference state $\|\psi_0\rangle$	SHO ground state $\|0\rangle$	$\|s, s\rangle$	
Phase space X	$\mathbb{R}^{2N}$	Sphere S^2	
SW kernel $\hat{\Delta}(\xi)$	$\int \frac{d^{2N}\boldsymbol{\xi}'}{\pi^2} \cdot e^{i\boldsymbol{\xi}'^T\boldsymbol{\Omega}\boldsymbol{\xi}} \hat{D}(\boldsymbol{\xi})$	$\sqrt{\frac{4\pi}{2j+1}} \sum_{l=0}^{2j} \sum_{m=-l}^{l} \hat{D}_{lm} Y^*_{lm}(\theta, \phi)$	
Symplectic form ω	$\sum_i dq_i \wedge dp_i$	$j \sin\theta \, d\theta \wedge d\phi$	

SW correspondence and the KKS symplectic form

	CV systems	Spin systems	Bipartite systems
Hilbert space H	$L^2(\mathbb{R}^N)$	$\mathbb{C}^{2j+1}$	$H_A \otimes H_B$
Dynamical group G	Heisenberg group H_{2N+1}	SU(2)	$G_A \times G_B$
Reference state $\|\psi_0\rangle$	SHO ground state $\|0\rangle$	$\|s, s\rangle$	$\|\psi_0\rangle_A \otimes \|\psi_0\rangle_B$
Phase space X	$\mathbb{R}^{2N}$	Sphere S^2	$X_A \times X_B$
SW kernel $\hat{\Delta}(\xi)$	$\int \frac{d^{2N}\boldsymbol{\xi}'}{\pi^2} \cdot e^{i\boldsymbol{\xi}'^T\boldsymbol{\Omega}\boldsymbol{\xi}} \hat{D}(\boldsymbol{\xi})$	$\sqrt{\frac{4\pi}{2j+1}} \sum_{l=0}^{2j} \sum_{m=-l}^{l} \hat{D}_{lm} Y_{lm}^*(\theta, \phi)$	$\hat{\Delta}_A(\xi_A) \otimes \hat{\Delta}_B(\xi_B)$
Symplectic form ω	$\sum_i dq_i \wedge dp_i$	$j \sin\theta \, d\theta \wedge d\phi$	$\omega_A + \omega_B$

Partial transposition in Symplectic geometry[6]

Theorem

Assume that there is an equivariant diffeomorphism from $X_{A,B}$ to a coadjoint orbit $G_{A,B} \cdot \mu_{A,B}$ sending the class of $|\psi_{A,B}\rangle$ to $\mu_{A,B}$, and $\langle \mu_B, \eta \rangle = \langle \psi_B | i T_{B*} \eta | \psi_B \rangle$ for all ξ in the Lie algebra of G_B, and that the complex conjugation of T_B with respect to a basis containing $|\psi_B\rangle$ induces an involutive automorphism on G_B, then $W_{\hat{\rho}^{T_B}}(\xi) = W_{\hat{\rho}}(\varphi(\xi))$, where φ is a separable symplectomorphism from (X, ω) to (X, ω_-).

[6] Yi-Ting Tu and Ray-Kuang Lee, in preparation

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And, using the same framework as in the CV case, we still has

Proposition

A state satisfies PPT iff it is valid with respect to ω_-.

[6] Yi-Ting Tu and Ray-Kuang Lee, in preparation

Summary

- PPT criterion: if $\hat{\rho}$ is separable, then $\hat{\rho}^{T_B}$ is positive (a valid quantum state).
- In our framework of describing states with geometric objects, PPT is equivalent to validity with respect to the symplectic form ω_-.
- Examples of this framework include Gaussian states, cat states, and the zero eigenstates of quadratures.
- Under some technical assumptions, this framework can be generalized to other quantum systems, including spin systems.